FB Mathematik
Thomas Markwig

Winterterm 2009/10, Set 7

Georges Francois

# Algebraic Geometry I

Due date: Monday, 14/12/2009, 10:00 Uhr

**Exercise 25:** Which of the following algebraic sets are isomorphic?

$$\mathbb{A}^1_{\mathbb{C}} \qquad Z(xy) \subseteq \mathbb{A}^2_{\mathbb{C}} \qquad Z(x^2+y^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$$

$$Z(x^2-y^3) \subseteq \mathbb{A}^2_{\mathbb{C}} \qquad Z(y-x^2, z-x^3) \subseteq \mathbb{A}^3_{\mathbb{C}}.$$

Hint, in principle you have to consider the coordinate rings and show whether they are isomorphic or not. But you can determine the isomorphisms on the geometric side, and for non-isomorphic sets you may perhaps argue by geometric properties which prevent the existence of an isomorphism.

**Exercise 26:** Which of the following claims is correct? Prove or give a counter example.

a. A morphism $\varphi : \mathbb{A}^2_K \longrightarrow \mathbb{A}^2_K$ is an isomorphism if there are $a, b, c, d, e, f \in K$ such that

$$\varphi(x, y) = (ax + by + e, cx + dy + f)$$

for all $(x, y) \in \mathbb{A}^2_K$ with

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

b. Every automorphism of $\mathbb{A}^2_K$ is of this form.

Hint, the map $\varphi^*$ on $K[x, y]$ must be an isomorphism as well, i.e. it is a coordinate change given by $x \mapsto f(x, y)$ and $y \mapsto g(x, y)$, and the inverse is of the same type. Try to see how that restricts $f$ and $g$.

**Exercise 27:**

a. Let $X \subseteq \mathbb{A}^n_K$ be an affine algebraic sets, $U \subseteq X$ open and $\varphi_1, \ldots, \varphi_m \in \mathcal{O}_X(U)$. Show that the map

$$\varphi : U \longrightarrow \mathbb{A}^m_K : p \mapsto \big(\varphi_1(p), \ldots, \varphi_m(p)\big)$$

is a morphism.

b. Let $\varphi : \mathbb{A}^n_K \longrightarrow \mathbb{A}^m_K$ be a morphism.

(a) Is the preimage $\varphi^{-1}(Y) \subseteq \mathbb{A}^n_K$ an algebraic set if $Y \subseteq \mathbb{A}^m_K$ is an algebraic set?

(b) Is the graph $\Gamma = \{(p, \varphi(p)) \in \mathbb{A}^{n+m}_K \mid p \in X\} \subseteq \mathbb{A}^{n+m}_K$ an algebraic set if $X \subseteq \mathbb{A}^n_K$ is an algebraic set?

Hint: a. It suffices to show that $\varphi$ is continous. Consider the preimage of a closed subset $Y = Z(I) = \bigcap_{f \in I} Z(f) \subseteq \mathbb{A}^m_K$ under $\varphi$ and show that it is closed. b.(b) Either look for a counter example among the algebraic sets you know, or try to find a set of polynomials whose zero locus Graph$(\varphi)$ is, taking the ideal of $X$ and the component functions of $\varphi$ into account.

**Exercise 28:** Let $K$ be an algebraically closed field and let $f \in K[x, y]$ be an irreducible polynomial of degree two. Show that $Z(f)$ is isomorphic to either $Z(y - x^2)$ or to $Z(xy - 1)$.

Hint, each polynomial $f$ of degree two can be written uniquely as $f = (x, y) \cdot A \cdot (x, y)^t + v \cdot (x, y)^t + d$ for some $2 \times 2$-Matrix $A$, some vector $v \in K^2$ and some constant $d$. Find a coordinate transformation as in Exercise 26 which transforms $f$ into one of the two above normal forms, and do so by distinguishing the two cases that $\mathrm{rank}(A) = 2$ respectively $\mathrm{rank}(A) = 1$.