FB Mathematik | Winterterm 2009/10, Set 8
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Algebraic Geometry I

Due date: Monday, 04/01/2010, 10:00 Uhr

Exercise 29: Let $\varphi : (X,\mathcal{F}) \longrightarrow (Y,\mathcal{G})$ be a morphism of affine varieties over an algebraically closed field K (as defined in Definition 3.1 of the lecture), and let $\varphi^* : \mathcal{G}(Y) \longrightarrow \mathcal{F}(X)$ be the pull-back on the global section. Which of the following statements is correct?

a. φ^* is injective if and only if φ is surjective.

b. φ^* is surjective if and only if φ is injective.

Either prove the statement or give a counterexample. Can you weaken incorrect statements so that they become correct?

Hint, you can right away reduce to the case where X and Y are affine algebraic sets. An interesting example in this context is the projection of the hyperbola $Z(xy-1)$ onto the x-axis.

Exercise 30: Let $(X,\mathcal{F})$ be a prevariety.

a. Show that X is a noetherian topological space.

b. Show that $(U,\mathcal{F}_{|U})$ is a prevariety for any open subset $U \subseteq X$.

Hint: a. cover X by finitely many affine varieties and cut a descending chain of closed subspaces down to the affine pieces. b. cover X by affine varieties $X_1,\ldots,X_k$ and use Proposition 2.7 for $U\cap X_i$ to cover it by affines.

Exercise 31: Let $(X,\mathcal{F})$ be a prevariety and let $Y \subseteq X$ be a closed subset. For an open subset $U \subset Y$ we define

$$\mathcal{G}(U) = \{g : U \longrightarrow K \mid \forall\, p \in U\ \exists\, p \in V \subseteq X \text{ open } \&\ f \in \mathcal{F}(V)\ :\ g \equiv f \text{ on } U\cap V\}.$$

a. Show that $\mathcal{G}$ together with the usual restriction maps is a sheaf of K-algebras, and thus $(Y,\mathcal{G})$ is a space with K-valued functions.

b. Show that $(Y,\mathcal{G})$ is actually a prevariety.

c. Show that $\mathcal{F}(V)/I_Y(V) \longrightarrow \mathcal{G}(V\cap Y) : \overline{f} \mapsto f_{|V\cap Y}$ is a monomorphism for any open subset $V \subseteq X$ where $I_Y(V) = \{f \in \mathcal{F}(V) \mid f(p) = 0\ \forall\, p \in V\cap Y\}$. Is this map always an isomorphism?

Hint: b. use an affine covering of X and cut it down to Y. In part c. you might want to consider $X = V = \mathbb{P}^1_{\mathbb{C}}$ and $Y = \{0,1\}$, or $X = \mathbb{A}^2_{\mathbb{C}}$, $Y = \mathbb{A}^1_{\mathbb{C}}$ and $V = X \setminus \{(0,0)\}$.

Exercise 32: Show that the prevariety $\mathbb{P}^1_K$ is a variety.

Hint, $\mathbb{P}^1_K$ is covered by two affine varieties X and Y, so that $\mathbb{P}^1_K \times \mathbb{P}^1_K$ is covered by the four affine varieties $X\times X$, $X\times Y$, $Y\times X$ and $Y\times Y$. Use this to show that $\Delta(\mathbb{P}^1_K)$ is closed in $\mathbb{P}^1_K \times \mathbb{P}^1_K$.