Recursive formulas for real enumerative invariants

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September 3, 2008

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1. Welschinger invariants of real Del Pezzo surfaces.

The Gromov-Witten (further on, shortly *GW*) theory is the key tool for the enumeration of complex algebraic curves of a given genus and a given homology class, lying in a given complex algebraic variety and matching the respective number of generic constraints.

The Welschinger invariants serve as a real analogue of the GW-invariants. As enumerative invariants, they are defined for real rational symplectic manifolds of dimension ≥ 4, and count real rational pseudoholomorphic curves in a given homology class. In the case of real Del Pezzo surfaces (whose complex structure is generic among tame almost complex structures in the underlying symplectic manifold), Welschinger invariants count real algebraic curves in a given linear system passing through respective number of generic points.

Let Σ be a real Del Pezzo surface with a non-empty real part. Given

- a real effective divisor $D \subset \Sigma$,
- a generic configuration ω of $(-DK_\Sigma - 1)$ points, containing $m \geq 0$ pairs of imaginary conjugate points and $(-DK_\Sigma - 1 - 2m)$ real points on a connected component $\Sigma_i \subset \mathbb{R}\Sigma$,

the *Welschinger invariant* $W_{m,\Sigma_i}(\Sigma, D)$ is defined as the number of real rational curves $C \in |D|$ on Σ, passing through the configuration ω, and counted with weights $w(C) = (-1)^{s(C)}$, where $s(C)$ is the number of the solitary real nodes of C. The number $W_{m,\Sigma_i}(\Sigma, D)$ does not depend on the choice of ω, but only on the parameters m, Σ_i, and D.

Further on, we consider only Welschinger invariants associated with the totally real configurations of points, i.e. $m = 0$, and the surfaces with a connected nonempty real point set (i.e., $b_0(\mathbb{R}\Sigma) = 1$), and thus, we simply write $W_0(\Sigma, D)$.

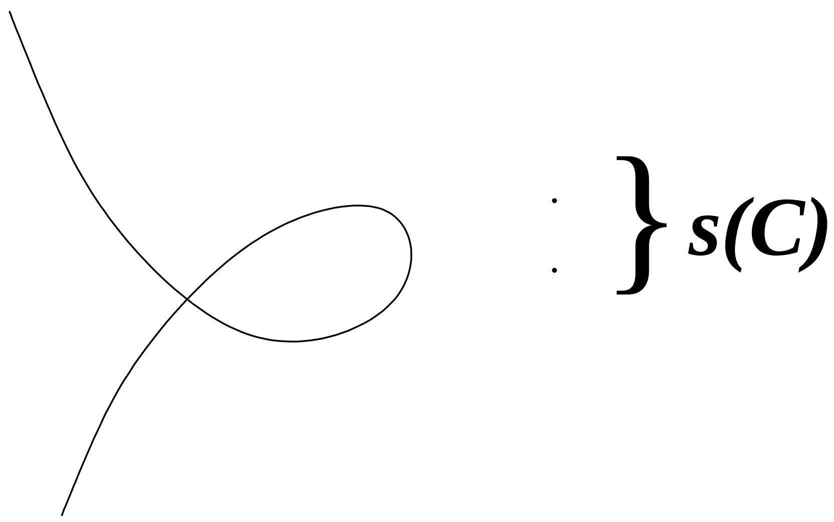


Figure 1: Welschinger sign of a real rational curve

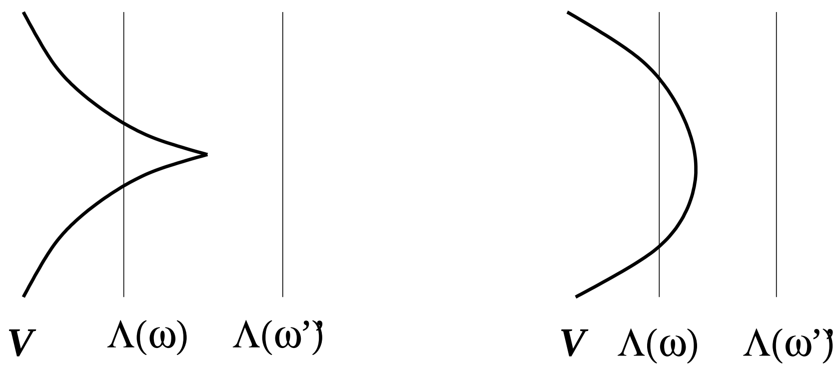


Figure 2: Bifurcations of $R_\omega(\Sigma, D)$

As an immediate enumerative application, we obtain the relation

$$|W_0(\Sigma, D)| \le R_\omega(\Sigma, D) \le GW_0(\Sigma, D) \ , \tag{1}$$

where $R_\omega(\Sigma, D)$ is the number of real rational curves $C \in |D|$, passing through the configuration ω, and $GW_0(\Sigma, D)$ is the respective genus zero GW-invariant (i.e., the number of complex rational curves $C \in |D|$, passing through the configuration ω).

Inequality (1) provides answers the following important questions:

Question 1. *Given a real Del Pezzo surface Σ, a real ample divisor $D \subset \Sigma$, and a generic conjugation invariant configuration ω of $-DK_\Sigma - 1$ points in Σ, does there exist a real rational curve $C \in |D|$, passing through ω?*

Question 2. *Assuming that $R_\omega(\Sigma, D)$ is always positive, how far is it from $GW_0(\Sigma, D)$?*

The known ways to compute Welschinger invariants are based on
- the tropical algebraic geometry (Mikhalkin, Itenberg, Kharlamov, Sh., Brugallé),
- the theory of open Gromov-Witten invariants (Solomon),
- the symplectic field theory (Welschinger).

We shall discuss the tropical approach which allows one to answer the above questions in a series of interesting cases, leads to Caporaso-Harris type recursive formulas for the Welschinger invariants, and reveals interesting phenomena in the tropical geometry, in particular, the existence of tropical enumerative invariants.

2. Welschinger invariants associated with the totally real configurations of points.

Example 1 (Itenberg, Kharlamov, Sh.) *For the projective plane,*

d	1	2	3	4	5	6
W_0	1	1	8	240	18264	2845440
GW_0	1	1	12	620	67304	26312976

One can see that these invariants are positive and grow very rapidly. Precise statements are as follows:

Theorem 1 (IKS) *Let Σ be a real toric Del Pezzo surface Σ with a nonempty real point set, or $\mathbb{P}^2_{k,l}$, $4 \le k+2l \le 5$, the plane, blown up at a generic configuration of k real and l pairs of imaginary conjugate points. Then, for any real ample divisor $D \subset \Sigma$,*

$$W_0(\Sigma, D) > 0 \ , \tag{2}$$

and

$$\lim_{n\to\infty} \frac{\log W_0(\Sigma, nD)}{n \log n} = \lim_{n\to\infty} \frac{\log GW_0(\Sigma, nD)}{n \log n} = -DK_\Sigma \ . \tag{3}$$

In particular, through any generic configuration of $-DK_\Sigma - 1$ real points on Σ, one can trace a real rational curve $C \in |D|$), and, moreover, the number of such curves is logarithmic equivalent to the number of all (complex) rational curves through the given points. So, among the main problems remains

Conjecture 1 *Relations (2) and (3) hold for any real Del Pezzo surface Σ with $b_0(\mathbb{R}\Sigma) = 1$ and for any real ample divisor $D \subset \Sigma$.*

Among other observed properties of $W_0(\Sigma, D)$ we mention the monotone behavior with respect to the divisor D and arithmetic relations to the Gromov-Witten invariants:

Theorem 2 *Let $\Sigma = \mathbb{P}^2$, or $\mathbb{P}^2_k$, $k = 1, 2, 3$, or $\Sigma = (\mathbb{P}^1)^2$ (i.e., a toric Del Pezzo surface with the standard real structure). Then*

- *(IKS) for any real ample divisors D, D' with effective $D - D'$,*
$$W_0(\Sigma, D) \geq W_0(\Sigma, D') \ , \tag{4}$$
and, furthermore,
$$W_0(\Sigma, D) > W_0(\Sigma, D') \quad as \quad p_a(D) > p_a(D') \ ; \tag{5}$$
- *(Mikhalkin) for a real ample divisor D,*
$$W_0(\Sigma, D) \equiv GW_0(\Sigma, D) \mod 4 \ . \tag{6}$$

Numerical experiments suggest the following

Conjecture 2

- *Inequalities (4) and (5) hold for any real Del Pezzo surface Σ with $b_0(\mathbb{R}\Sigma) = 1$.*
- *Congruence (6) holds for any real Del Pezzo M-surface Σ (i.e., satisfying $b_*(\Sigma, \mathbb{Z}_2) = b_*(\mathbb{R}\Sigma, \mathbb{Z}_2)$) and any real ample divisor $D \subset \Sigma$.*

Question 3. *Is there any congruence, relating W_0 and GW_0, for $(M - i)$-surfaces? What powers of 2 divide $W_0(\Sigma, D)$? $GW_0(\Sigma, D)$?*

3. The tropical enumerative geometry in the plane.

A parameterized plane tropical curve (PPT-curve) is a triple $T = (\overline{\Gamma}, w, h)$, where Γ is a finite graph without divalent vertices, $w : \Gamma^1 \to \mathbb{Z}$ a positive weight function defined on the set of the edges Γ^1 of $\overline{\Gamma}$, and $h : \Gamma \to \mathbb{R}^2$ a proper continuous map defined on Γ, the complement to the set of univalent vertices, such that

- h takes any edge of Γ injectively to a straight line with a rational slope,
- for any vertex v of Γ, one has the balancing condition

$$\sum_{v \in e \in \Gamma^1} w(e) \cdot \boldsymbol{u}_v(e) = 0 \ ,$$

  where $\boldsymbol{u}_v(e)$ is the primitive integral vector emanating from $h(v)$ and directing $h(e)$, and the nondegeneracy condition

$$\mathrm{Span}\{\boldsymbol{u}_v(e) \ : \ v \in e \in \Gamma^1\} = \mathbb{R}^2 \ .$$

The *degree* of T is the unordered sequence of the vectors

$$\{w(e) \cdot \boldsymbol{u}_v(e) \ : \ v \in e \in \Gamma^1_\infty, \ v \in \Gamma^0\} \ ,$$

Γ^1_∞ being the set of ends Γ (edges merging to univalent vertices. The degree uniquely determines the *Newton polygon* of T. If Γ is connected, T is said to be *irreducible.* The *genus* of an irreducible curve is $g(T) = b_1(\Gamma)$.

If Γ is trivalent graph, and $w(e) = 1$ for all ends e, the curve T is called *simple.*

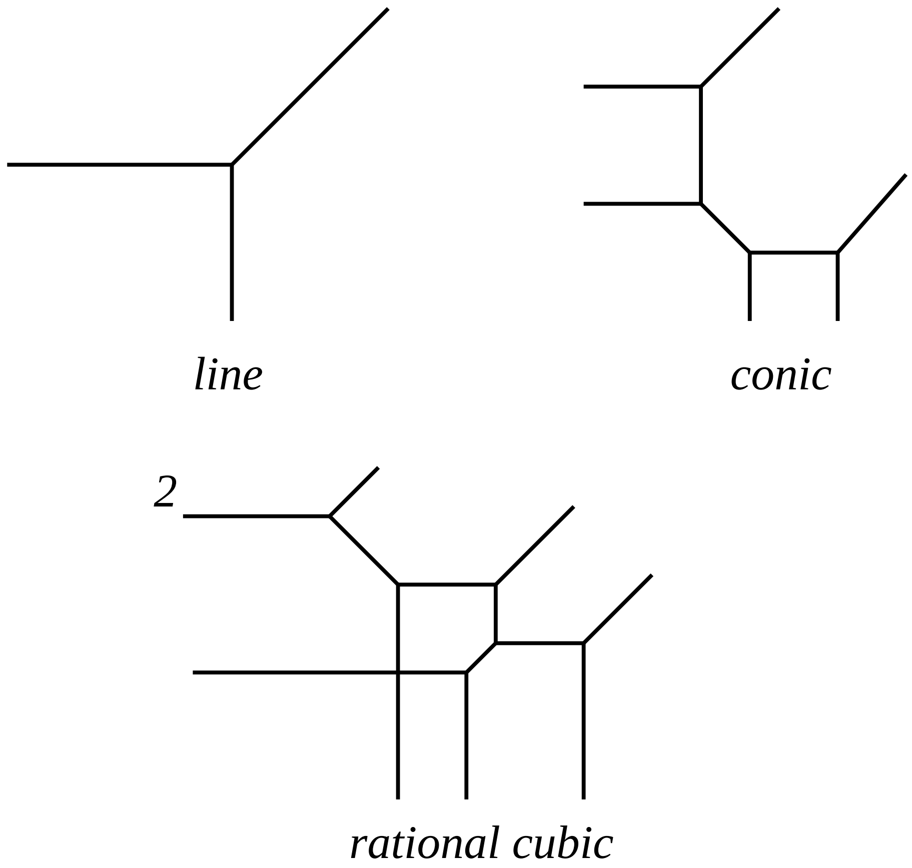


Figure 3: Plane tropical curve

4. Correspondence theorems.

Theorem 3 (Mikhalkin) *Let $\Delta \subset \mathbb{R}^2$ be a nongegenerate convex lattice polygon, $\Sigma = \mathrm{Tor}(\Delta)$ an associated toric surface, $\mathcal{L}_\Delta$ the tautological line bundle.*

- *the number of irreducible curves $C \in |\mathcal{L}_\Delta|$ of genus g, passing through a generic configuration of $-DK_\Sigma - 1 + g$ points in Σ, equals to the number of simple plane irreducible PPT-curves T of genus g with Newton polygon Δ, passing through a generic configuration of $-DK_\Sigma - 1 + g$ points in $\mathbb{R}^2$ and counted with certain integral positive multiplicities $M(T)$.*

- *If Σ is a toric Del Pezzo surface with the standard real structure, then $W_0(\Sigma, \mathcal{L}_\Delta)$ equals to the number of simple irreducible rational curves T, passing through a generic configuration of $-DK_\Sigma - 1$ points in $\mathbb{R}^2$ and counted with certain multiplicities $W(T)$ (± 1 for T with odd-valued weight function w, and 0 otherwise).*

There are similar correspondence theorems for computing W_m with $m > 0$ (Sh). All they serve as a key tool in the proof of the known properties of Welschinger invariants, but, on the other hand, are restricted to the toric case.

The correspondence: consider the algebraic enumerative problem over the field of complex Puiseux series $\mathbb{K} = \bigcup \mathbb{C}((t^{1/k}))$ with the non-Archimedean valuation $\mathrm{Val}(\sum_r a_r t^r) = -\min\{r \ : \ a_r \neq 0\}$. Then

$$\mathit{Closure}\left(\mathrm{Val}\left\{(x,y) \in (\mathbb{K}^*)^2 \ : \ \sum_{(i,j)\in\Delta} A_{ij} x^i y^j = 0\right\}\right)$$

is (the image of) a plane tropical curve with Newton polygon Δ.

5. The Caporaso-Harris formula for Welschinger invariants.

Let $\Sigma = \mathbb{P}^2$, $(\mathbb{P}^1)^2$, or $\mathbb{P}^2_{k,l}$, $1 \le k + 2l \le 5$. Let $E \subset \Sigma$ be a smooth real rational curve, avoiding imaginary exceptional divisors, with the minimal possible value of E^2. Denote by $A(\Sigma, E) \subset \mathrm{Pic}(\Sigma)$ the semigroup, which contains 0 and is generated by the irreducible real effective divisors, crossing E. Introduce an operation $S_E : A(\Sigma, E) \to A(\Sigma, E)$, letting $S_E(D) = D - E$ if $D - E \in A(\Sigma, E)$, and $S_E(D) = 0$ otherwise. The classes $D \in S_E^{-1}(0) \subset A(\Sigma)$ are called initial.

Introduce the semigroup $\mathbb{Z}_+^{\infty,odd}$ of integral nonnegative vectors with finite norms

$$|\alpha| = \sum_j \alpha_j, \quad J\alpha = \sum_j (2j-1)\alpha_j, \quad \alpha = (\alpha_1, \alpha_2, ...) \in \mathbb{Z}_+^{\infty,odd} ,$$

and put

$$\widetilde{A}(\Sigma, E) = \{(D, \alpha, \beta) \in A(\Sigma, E) \oplus (\mathbb{Z}_+^{\infty,odd})^2 \ : \ J(\alpha + \beta) = DE\} .$$

Let a function $W : \widetilde{A}(\Sigma, E) \to \mathbb{Z}$ satisfy

$$W(D, \alpha, \beta) = \sum_{\substack{k \ge 1 \\ \beta_k > 0}} W(D, \alpha + e_k, \beta - e_k)$$

$$+ \sum \binom{\alpha}{\alpha^{(1)}, ..., \alpha^{(m)}} \frac{n!}{n_1!...n_m!} \prod_{i=1}^m \beta_{k_i} W(D^{(i)}, \alpha^{(i)}, \beta^{(i)}) , \tag{7}$$

where the second sum is taken over all splittings in $\widetilde{A}(\Sigma)$ (factorized by possible permutations)

$$(S_E(D), \alpha', \beta') = \sum_{i=1}^m (D^{(i)}, \alpha^{(i)}, \beta^{(i)}) ,$$

satisfying the following conditions

$$\alpha' \le \alpha, \quad \beta' = \beta + \sum_{i=1}^m e_{k_i}, \quad \beta^{(i)}_{k_i} \ge 1, \quad J(\alpha' + \beta') = S_E(D) \cdot E ,$$

and the remaining items are defined by

$$\binom{\alpha}{\alpha^{(1)}, ..., \alpha^{(m)}} = \prod_{k \ge 1} \frac{\alpha_k!}{\alpha_k^{(1)}!...\alpha_k^{(m)}!(\alpha_k - \alpha'_k)!} ,$$

$$n = -(K_\Sigma + E)D + |\beta| - 1, \quad n_i = -(K_\Sigma + E)D^{(i)} + |\beta^{(i)}| - 1, \ i = 1, ..., m .$$

Theorem 4 (IKS) *The function* $W : \widetilde{A}(\Sigma, E) \to \mathbb{Z}$ *defined by formula (7), is uniquely determined by the initial values*

$$W(D, \alpha, 0) = \begin{cases} 1, & \textit{if the generic member of } D \textit{ is irreducible,} \\ 0, & \textit{if the generic member of } D \textit{ is reducible,} \end{cases}$$

where $S_E(D) = 0$*, and satisfies*

$$W(D, 0, (DE)) = W_0(\Sigma, D) \quad \textit{for all } D \in A(\Sigma, E) \ .$$

Remark 2 *This theorem provides the positivity and the asymptotics of* $W_0(\Sigma, D)$ *in case of the real non-toric Del Pezzo surfaces* $\mathbb{P}^2_{k,l}$*,* $4 \leq k + 2l \leq 5$*.*

Idea of the proof.

- First, we prove a Caporaso-Harris type formula for the numbers, which count specific PPT-curves T with certain multiplicities $M(T)$, and then show that it coincides with the Caposaso-Harris-Vakil formula for the absolute and relative genus zero Gromov-Witten invariants of Σ (relative invariants count rational curves which satisfy imposed tangency conditions with respect to E).

- Using a suitably modified patchworking construction, we assign to each PPT-curve T a set of $M(T)$ algebraic curves $C \subset \Sigma$ in count so that all the sets are disjoint.

- Altogether this provides us with a kind of a correspondence theorem.

- Next, we select real curves C in the above sets, count them with the weights $(-1)^{s(C)}$, and show that these numbers are equal to the numbers of the considered PPT-curves with certain multiplicities $W(T)$.

- Finally, we prove the aforementioned Caporaso-Harris type formula for the latter numbers.

6. Moduli spaces of plane tropical curves and tropical enumerative invariants.

One can view the discussed complex enumerative invariants as degrees of evaluation maps from suitable moduli spaces of marked PPT-curves to Euclidean spaces (Mikhalkin, Gathmann, Markwig).

For, we modify the definition of a tropical curve, requiring that Γ is a metric graph, whose ends have infinite length, making $h : \Gamma \to \mathbb{R}^2$ to be $\mathbb{Z}$-linear on each edge and non-vanishing on the ends, leaving the balancing condition, and removing the non-degeneracy condition for vertices of Γ. An *n-marked* plane tropical curve is a quadruple $(\overline{\Gamma}, w, G, h)$, where $(\overline{\Gamma}, w, h)$ is a (parameterized) plane tropical curve, $G \subset \overline{\Gamma}$ an ordered sequence of n distinct points. An isomorphism of marked tropical curves $(\overline{\Gamma}, w, G, h)$ and $(\overline{\Gamma}', w', G', h')$ is an isometry $\varphi : \Gamma \to \Gamma'$ such that $h'\varphi = h$, $w'\varphi = w$, and $G' = \varphi(G)$.

Given a convex lattice polygon $\Delta \subset \mathbb{R}^2$ and $g \le |\mathrm{Int}(\Delta) \cap \mathbb{Z}^2|$, put $n = |\partial\Delta \cap \mathbb{Z}^2| - 1 + g$. Then the moduli space $\mathcal{M}_{g,n}(\Delta)$ of simple n-marked plane tropical curves with Newton polygon Δ such that $G \subset \Gamma$ is the union of finitely many open $2n$-dimensional polyhedra in some $\mathbb{R}^N$. Its closure $\overline{\mathcal{M}}_{g,n}(\Delta)$ is a closed finite pure-dimensional polyhedral complex. Assigning weight 1 to each top-dimensional face, we obtain a tropical varianty, i.e., the balancing condition along each (top$-$1)-dimensional face. This implies, in particular, that the evaluation map $ev : \overline{\mathcal{M}}_{g,n}(\Delta) \to \mathbb{R}^{2n}$ (which is a restriction of a $\mathbb{Z}$-linear map $F : \mathbb{R}^N \to \mathbb{R}^{2n}$) has a well defined degree

$$\deg(ev) = \sum_{T \in ev^{-1}(\overline{\boldsymbol{x}})} |\det(d(ev)_T)|, \quad \overline{\boldsymbol{x}} \in \mathbb{R}^{2n} \text{ generic} .$$

Furthermore, $|\det(d(ev)_T)| = M(T)$ and

$$\deg(ev) = GW_g(\Sigma, \mathcal{L}_\Delta), \quad \Sigma = \mathrm{Tor}(\Delta) .$$

Welschinger invariants can be read off the moduli space of tropical curves as well. Assume that $\Sigma = \mathrm{Tor}(\Delta)$ is the real toric Del Pezzo surface with the standard real structure, $n = |\partial\Delta \cap \mathbb{Z}^2| - 1$. Then

$$W_0(\Sigma, \mathcal{L}_\Delta) = \sum_{T \in ev^{-1}(\overline{\boldsymbol{x}})} W(T), \quad \overline{\boldsymbol{x}} \in \mathbb{R}^{2n} \text{ generic} .$$

More geometrically, take $M^{(0)} \subset \overline{\mathcal{M}}_{0,n}(\Delta)$, the closure of the union of those top-dimensional cells of $\mathcal{M}_{0,n}(\Delta)$ on which $|\det(d(ev))| \neq 0$. Notice, that

$$ev : (M^{(0)}, \partial_\infty M^{(0)}) \to (\mathbb{R}^{2n}, \partial_\infty \mathbb{R}^{2n})$$

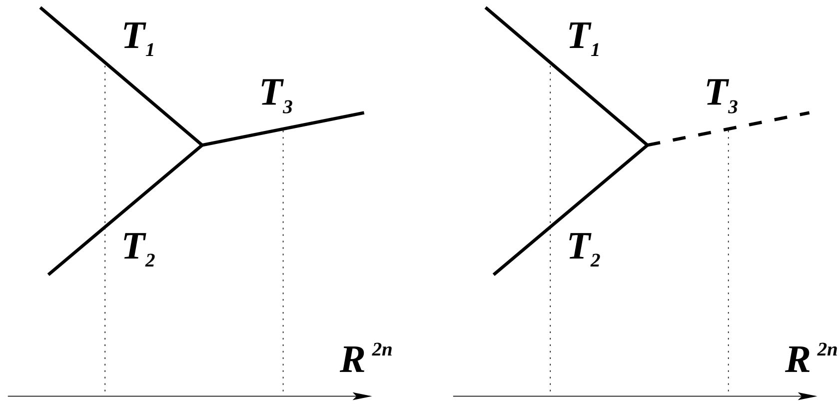
is well defined. Then we take $M^{(0)}_W \subset M^{(0)}$, the union of the cells, corresponding to PPT-curves with odd-valued weight function w, and orient each facet $\mathcal{C}$ of $M^{(0)}_W$, using the map ev and the sign $W(T)$, $T \in \mathcal{C}$.

Claim. (IKS) *$M^{(0)}_W$ is a relative cycle in $(M^{(0)}, \partial_\infty M^{(0)})$, and*

$$ev_*[M^{(0)}_W] = W_0(\Sigma, \mathcal{L}_\Delta) \in H_{2n}(\mathbb{R}^{2n}, \partial_\infty \mathbb{R}^{2n}) .$$

Question 4. *What is the meaning of $W_0(\Sigma, \mathcal{L}_\Delta)$ from the tropical intersection theory point of view?*

Question 5. *What is $H_{2n}(M, \partial_\infty M)$? What is the meaning of the classes outside* $\mathrm{Span}\{[M^{(0)}_W], [M^{(0)}_{GW}]\}$ *?*



$$ev : \overline{\mathcal{M}}_{g,n}(\Delta) \to \mathbb{R}^{2n} \qquad\qquad ev : M_W^{(g)} \to \mathbb{R}^{2n}$$

Figure 4: Moduli space of plane tropical curves and evaluation map

$$\begin{cases} M(T_1) + M(T_2) = M(T_3) \\ W(T) \equiv M(T) \mod 4 \\ W(T) = 0,\ \pm 1 \end{cases} \quad \Longrightarrow \quad W(T_1) + W(T_2) = W(T_3)$$

First step out. Consider tropical curves of a *positive* genus g. In the same manner as above, construct $M^{(g)}_W \subset M^{(g)} \subset \overline{\mathcal{M}}_{g,n}(\Delta)$.

Claim. (IKS) *The map* $ev : (M^{(g)}, \partial_\infty M^{(g)}) \to (\mathbb{R}^{2n}, \partial_\infty \mathbb{R}^{2n})$ *is well defined,* $M^{(g)}_W$ *is a relative cycle in* $(M^{(g)}, \partial_\infty M^{(g)})$, *and*

$$ev_*[M^{(g)}_W] = W_0^{g,trop}(\Sigma, \mathcal{L}_\Delta) \in H_{2n}(\mathbb{R}^{2n}, \partial\mathbb{R}^{2n}) \ ,$$

$$W_0^{g,trop}(\Sigma, \mathcal{L}_\Delta) = \sum_{T \in ev^{-1}(\overline{\boldsymbol{x}})} W(T), \quad \overline{\boldsymbol{x}} \in \mathbb{R}^{2n} \ \textit{generic} \ .$$

Remark 3 $W_0^{g,trop}(\Sigma, \mathcal{L}_\Delta)$ *equals the number of irreducible real algebraic curves* $C \in |\mathcal{L}_\Delta|$ *of genus* g*, passing through a* **tropically** *generic configuration* ω *of* n *points in* $\mathbb{R}\Sigma$ *and counted with signs* $(-1)^{s(C)}$. *However this number of real algebraic curves* **changes** *when varying* ω *in the space of all generic configurations of real points on* Σ.

Claim. (IKS) *For any real toric Del Pezzo surface* Σ *with the standard real structure, any real ample divisor* $D \subset \Sigma$*, and any integer* $0 < g \le D(D + K_\Sigma)/2 + 1$,

$$W_0^{g,trop}(\Sigma, D) > 0 \ .$$

Furthermore,

$$\lim_{k\to\infty} \frac{\log W_0^{g(k),trop}(\Sigma, kD)}{k \log k} = \lim_{k\to\infty} \frac{\log GW_{g(k)}(\Sigma, kD)}{k \log k} = -DK_\Sigma + g_0 \ ,$$

where

$$\lim_{k\to\infty} \frac{g(k)}{k} = g_0, \quad 0 \le g_0 \le \frac{D(D+K_\Sigma)}{2} + 1 \ .$$

Second step out. Perform the preceding construction for the case of arbitrary nondegenerate convex lattice polygon $\Delta \subset \mathbb{R}^2$:

$$M_W^{(0)} \subset M^{(0)} \subset \overline{\mathcal{M}}_{0,n}(\Delta) \ .$$

Again

Claim. *The map $ev : (M^{(0)}, \partial_\infty M^{(0)}) \to (\mathbb{R}^{2n}, \partial_\infty \mathbb{R}^{2n})$ is well defined, $M_W^{(g)}$ is a relative cycle in $(M^{(0)}, \partial_\infty M^{(0)})$, and*

$$ev_*[M_W^{(0)}] = W_0^{0,trop}(\Sigma, \mathcal{L}_\Delta) \in H_{2n}(\mathbb{R}^{2n}, \partial \mathbb{R}^{2n}) \ ,$$

$$W_0^{0,trop}(\Sigma, \mathcal{L}_\Delta) = \sum_{T \in ev^{-1}(\overline{\boldsymbol{x}})} W(T), \quad \overline{\boldsymbol{x}} \in \mathbb{R}^{2n} \ \textit{generic} \ .$$

Remark 4 $W_0^{0,trop}(\Sigma, \mathcal{L}_\Delta)$ *equals the number of irreducible real rational algebraic curves C in the linear system $|\mathcal{L}_\Delta|$ on $\Sigma = \mathrm{Tor}(\Delta)$, passing through a* **tropically** *generic configuration ω of n points in $\mathbb{R}\Sigma$ and counted with signs $(-1)^{s(C)}$. However this number of real algebraic curves* **changes** *when varying ω in the space of all generic configurations of real points on Σ.*

Claim. *Assume that (up to $SL(\mathbb{Z}, 2)$-action) all the intersections $\Delta \cap \{x = s\}$, $s \in \mathbb{Z}$, are lattice segments (or empty). Then*

$$W_0^{0,trop}(\Sigma, \mathcal{L}_\Delta) > 0 \ .$$

Furthermore,

$$\lim_{k \to \infty} \frac{\log W_0^{0,trop}(\Sigma, \mathcal{L}_{k\Delta})}{k \log k} = \lim_{k \to \infty} \frac{\log GW_0(\Sigma, \mathcal{L}_{k\Delta})}{k \log k} = -c_1(\mathcal{L}_\Delta) \cdot K_\Sigma \ .$$

Third step out. Now consider the PPT-curves which have a given degree, i.e., a given Newton polygon Δ and a given distribution of positive integral weights on the ends. Furthermore, extend $h : \Gamma \to \mathbb{R}^2$ up to $\overline{h} : \overline{\Gamma} \to \mathrm{Tor}^{trop}(\Delta) \simeq \Delta$ and allow the marked points to be univalent vertices of $\overline{\Gamma}$.

The preceding procedure goes smoothly in this generality, allowing one to define real tropical enumerative invariants $W_0^{g,trop}(\Sigma, \mathcal{L}_\Delta, \alpha, \beta)$ for any genus $g \geq 0$ and any weight distributions of ends: α for ends with marked univalent vertex, β for free ends.

Remark 5 *In the case of toric Del Pezzo surfaces with the standard real structure, the numbers $W(D, \alpha, \beta)$ are the tropical enumerative invariants of the above sort.*

Question 6. *Are the numbers $W(D, \alpha, \beta)$ tropical enumerative invariants for the remaining surfaces $\mathbb{P}^2_{k,l}$, $k + 2l \leq 5$, $l > 0$?*

Fourth step out: $W_m(\Sigma, D)$, $m > 0$, as tropical enumerative invariants.

Notice that the valuation images of conjugate points coincide in $\mathbb{R}^2$. Thus, the set G of marked points of the corresponding PPT-curves $(\overline{\Gamma}, w, G, h)$ consists of r separate points and m pairs of points, where $r + 2m = n = -c_1(\mathcal{L}_\Delta)K_\Sigma - 1$, such that each pair is mapped to one point, and the points of a pair may collide at a vertex of Γ.

One can construct the moduli space $\overline{\mathcal{M}}_{0,(r,m)}(\Delta) \subset \mathbb{R}^N$ of such (r, m)-marked PPT-curves (tropical variety?), and define the evaluation map onto $\mathbb{R}^{2(r+m)}$ with the correctly defined degree

$$\deg(ev) = \sum_{T \in (ev)^{-1}(\overline{\boldsymbol{x}})} M(T) = GW_0(\Sigma, \mathcal{L}_\Delta), \quad \overline{\boldsymbol{x}} \in \mathbb{R}^{2(r+m)} \text{ generic },$$

and satisfying

$$\sum_{T \in (ev)^{-1}(\overline{\boldsymbol{x}})} W_m(T) = W_0(\Sigma, \mathcal{L}_\Delta), \quad \overline{\boldsymbol{x}} \in \mathbb{R}^{2(r+m)} \text{ generic } .$$

Furthermore, one can define a polyhedral subcomplex $M \subset \overline{\mathcal{M}}_{0,(r,m)}(\Delta)$ such that $ev(\partial_\infty M) \subset \partial_\infty \mathbb{R}^{2(r+m)}$, and then construct the chain $M_W = \sum W_m(T) \cdot \mathcal{C}$, $\mathcal{C}$ running over the facets of M.

Fact: *The above chain M_W is not a (relative) cycle in $(M, \partial_\infty M)$.*

7. WDVV-equation. The GW-invariants of Del Pezzo surfaces can be computed recursively, using the WDVV equation (system of equations). Namely, one can define the (genus zero) Gromov-Witten potential, which in the planar case turns into

$$\Phi(t_0,t_1,t_2)=\sum_{d,n\ge 0}\frac{GW_{0,d,n}((c_0t_0+c_1t_1+c_2t_2)^{\otimes n})}{n!}=\frac{t_0t_1^2}{2}+\sum_{d\ge 1}\frac{N_dt_2^{3d-1}e^{dt_1}}{(3d-1)!}$$

(here $c_0, c-1, c-2$ are the additive generators of $H^*(\mathbb{P}^2,\mathbb{Z})$), and the WDVV equations are

$$\sum_{i,j=0,1,2}\partial_a\partial_b\partial_i\Phi\cdot g^{ij}\cdot\partial_c\partial_d\partial_j\Phi=\sum_{i,j=0,1,2}\partial_a\partial_c\partial_i\Phi\cdot g^{ij}\cdot\partial_b\partial_d\partial_j\Phi\ ,$$

$a,b,c,d\in\{0,1,2\}$, $(g^{ij})=(c_i\cup c_j[\mathbb{P}^2])^{-1}$. This, in particular, implies the Kontsevich formula

$$N_d=\sum_{k+l=d}k^2lN_kN_l\left(l\binom{3d-4}{3k-2}-k\binom{3d-4}{3k-1}\right)\ .$$

Solomon [11, 12] defined open (genus zero) Gromov-Witten invariants $OGW_{d,k,l}:H^*_{conj}(X,\mathbb{Q})^{\otimes l}\to\mathbb{Q}$, X a real symplectic manifold, which in the planar case form an open Gromov-Witten potential

$$\Omega(t_0,t_1,t_2,u)=\sum_{d=2d_1,k,l}\frac{u^k}{k!l!}OGW_{d,k,l}((c_0t_0+c_1t_1+c_2t_2))^{\otimes l})$$

$$+\sqrt{-1}\sum_{d=2d_1+1,k,l}\frac{u^k}{k!l!}OGW_{d,k,l}((c_0t_0+c_1t_1+c_2t_2))^{\otimes l})$$

$$=u+\sum_{k+2l=6d-1}\frac{u^kt_2^le^{t_1d/2}}{k!l!}2^{1-l}(\sqrt{-1})^dW_{d,k,l}\ ,$$

satisfying the following analogues of WDVV equations

$$\sum_{i,j}\partial_a\partial_b\partial_i\Phi\cdot g^{ij}\cdot\partial_j\partial_c\Omega+\partial_a\partial_b\Omega\cdot\partial_u\partial_c\Omega=\sum_{i,j}\partial_c\partial_b\partial_i\Phi\cdot g^{ij}\cdot\partial_j\partial_a\Omega+\partial_c\partial_b\Omega\cdot\partial_u\partial_a\Omega\ ,$$

and

$$\sum_{i,j}\partial_a\partial_b\partial_i\Omega\cdot g^{ij}\cdot\partial_j\partial_u\Omega+\partial_a\partial_b\Omega\cdot\partial_u^2\Omega=\partial_a\partial_u\Omega\cdot\partial_b\partial_u\Omega\ ,$$

$a,b,c\in\{0,1,2\}$.

These equations lead to the following analogues of the Kontsevich formula:

(1) for $k \geq 2$,

$$W_{d,k} = \sum_{\substack{2d_1+d_2=d \\ d_1,d_2>0 \\ 3d_1-1\leq k}} (-1)^{d_1} 2^{3d_1-2} \left[d_2 \binom{k-2}{3d_1-2} - 2\binom{k-2}{3d_1-1} \right] d_1^2 d_2 N_{d_1} W_{d_2,k-3d_1}$$

$$+ \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0 \\ k_1+k_2=k-1 \\ k_1,k_2\geq 0}} \binom{k-2}{k_1} \left[d_2 \binom{3d-2k-1}{3d_1-2k_1-2} - d_1 \binom{3d-2k-1}{3d_1-2k_1-1} \right] W_{g_1,k_1} W_{d_2,k_2}$$

and

(2) for $k \geq 1$, $3d-2k-1 \geq 1$,

$$W_{d,k} = (-1)^{d/2+1} 2^{3d_2-4} d^2 N_{d/2} \delta_{k,3d/2-1}$$

$$- \sum_{\substack{2d_1+d_2=d \\ d_1,d_2>0 \\ 3d_1\leq k}} (-1)^{d_1} \binom{k-1}{3d_1-1} 2^{3d_1-2} d_1^3 d_2 N_{d_1} W_{d_1,k-3d_1}$$

$$+ \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0 \\ k_1+k_2=k-1 \\ k_1,k_2\geq 0}} \binom{k-1}{k_1} \left[d_2 \binom{3d-2k-2}{3d_1-2k_1-2} - d_1 \binom{3d-2k-2}{3d_1-2k_1-1} \right] d_1 W_{d_1,k_1} W_{d_2,k_2} \ .$$

The above sequences of formulas uniquely determine all the invariants $W_{d,k} := W_k(\mathbb{P}^2, d)$ from the initial value $W_{1,0} = 1$.

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