Fachbereich Mathematik
Dr. Thomas Keilen

Summer Semester 2004, Set 2

Computer Algebra

Due date: Monday, 17/05/2004, 10h00

Exercise 4: Let R be a unique factorisation domain and $S \subset R$ a multiplicatively closed subset.

a. Suppose that S is saturated (i.e. $s \cdot t \in S$ if and only if $s, t \in S$). Show that $f \in R$ is irreducible in $S^{-1}R$ if and only if $f = u \cdot g$ where $g \notin S$ is irreducible in R and $u \in S$.

b. Show that $S^{-1}R$ is a unique factorisation domain.

Hint, replacing S by its saturation in (b) we may assume that S is saturated – compare Commutative Algebra Exercise 10 and see Atyah-Macdonald, Chapter 3 Exercise 7.

Exercise 5: For two monomials $\underline{x}^\alpha, \underline{x}^\beta \in K[\underline{x}]$ we define

$$\gcd\left(\underline{x}^\alpha, \underline{x}^\beta\right) = x_1^{\min(\alpha_1,\beta_1)} \cdots x_n^{\min(\alpha_n,\beta_n)} \quad \text{and} \quad \operatorname{lcm}\left(\underline{x}^\alpha, \underline{x}^\beta\right) = x_1^{\max(\alpha_1,\beta_1)} \cdots x_n^{\max(\alpha_n,\beta_n)},$$

the greatest common divisor respectively the lowest common multiple of the two monomials, and they obviously satisfy the usual properties of gcd respectively lcm. Let $I = \langle \underline{x}^{\alpha_i} \mid i = 1, \ldots, k \rangle$ and $J = \langle \underline{x}^{\beta_j} \mid j = 1, \ldots, l \rangle$ be two monomial ideals in $K[\underline{x}]$ and let $\underline{x}^\gamma \in K[\underline{x}]$ be a monomial. Show that

a. $I \cap J = \left\langle \operatorname{lcm}\left(\underline{x}^{\alpha_i}, \underline{x}^{\beta_j}\right) \mid i = 1, \ldots, k;\ j = 1, \ldots, l \right\rangle$, and

b. $I : \underline{x}^\gamma = \left\langle \frac{\operatorname{lcm}\left(\underline{x}^{\alpha_i}, \underline{x}^\gamma\right)}{\underline{x}^\gamma} \;\middle|\; i = 1, \ldots, k \right\rangle$.

In particular, $I \cap J$ and $I : \underline{x}^\gamma$ are monomial ideals again.

Exercise 6: Let $>$ be a local ordering on $\operatorname{Mon}(x_1, \ldots, x_n)$. Show that

$$K(y_1, \ldots, y_m)[x_1, \ldots, x_n]_> = K[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\langle x_1, \ldots, x_n \rangle}.$$

Exercise 7: Give one possible realization of the following rings within SINGULAR:

a. $\mathbb{Q}[x, y, z]$,

b. $\mathbb{F}_5[x, y, z]$,

c. $\mathbb{Q}[x, y, z]/\langle x^5 + y^3 + z^2 \rangle$,

d. $\mathbb{Q}(i)[x, y]$, where i is the imaginary unit,

e. $\mathbb{F}_{27}[x_1, \ldots, x_{10}]_{\langle x_1, \ldots, x_{10} \rangle}$,

f. $\mathbb{F}_{32003}[x, y, z]_{\langle x,y,z \rangle}/\langle x^5 + y^3 + z^2, xy \rangle$,

g. $\mathbb{Q}(t)[x, y, z]$,

h. $\left(\mathbb{Q}[t]/\langle t^3 + t^2 + 1 \rangle\right)[x, y, z]_{\langle x,y,z \rangle}$,

i. $\mathbb{Q}[x, y, z]_{\langle x,y \rangle}$.