Fachbereich Mathematik
Dr. Thomas Keilen

Summer Semester 2004, Set 3

Computer Algebra

Due date: Monday, 24/05/2004, 10h00

Die SINGULAR-Aufgaben brauchen erst am Dienstag, den 25. Mai, um 18:00 Uhr abgegeben zu werden. Schreibt die Prozeduren bitte in eine Bibliothek, die Euren Namen trägt, schickt Cristina die Bibliothek als E-Mail und legt einen Ausdruck der Prozeduren in Ihr Fach.

Exercise 8: We define the *degree lexicographical ordering* $>_{Dp}$ on Mon_n by

$$\underline{x}^\alpha >_{Wp} \underline{x}^\beta \iff |\alpha| > |\beta| \text{ or } \big(|\alpha| = |\beta| \text{ and } \exists\, k\, :\, \alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k\big).$$

Show that the orderings $>_{lp}$, $>_{Dp}$ and $>_{dp}$ are described by the following characterising properties. Let $>$ be a monomial ordering on Mon_n, then:

a. $> = >_{lp}$ if and only if $>$ is an elimination ordering for $\{x_1, \ldots, x_k\}$ for all $k = 1, \ldots, n-1$, i. e. if $\text{lm}(f) \in K[x_{k+1}, \ldots, x_n]$ implies $f \in K[x_{k+1}, \ldots, x_n]$.

b. $> = >_{Dp}$ if and only if $>$ is a degree ordering and for any *homogeneous* $f \in K[\underline{x}]$ with $\text{lm}(f) \in K[x_k, \ldots, x_n]$ we have $f \in K[x_k, \ldots, x_n]$, $k = 1, \ldots, n$.

c. $> = >_{dp}$ if and only if $>$ is a degree ordering and for any *homogeneous* $f \in K[\underline{x}]$ with $\text{lm}(f) \in \langle x_k, \ldots, x_n \rangle$ we have $f \in \langle x_k, \ldots, x_n \rangle$, $k = 1, \ldots, n$.

Exercise 9: Let $I \trianglelefteq K[\underline{x}]$ be a homogeneous ideal in the polynomial ring. Show that the degree reverse lexicographical ordering $>_{dp}$ satisfies the properties:

$$L\big(I + x_n^d\big) = L(I) + \big\langle x_n^d \big\rangle \qquad \text{and} \qquad L\big(I : x_n^d\big) = L(I) : x_n^d \qquad \text{for any } d \geq 1.$$

Note, since I is homogeneous we only have to consider homogeneous polynomials – e. g. $L(I) = \langle \text{lm}(f) \mid f \in I, f \text{ homogeneous} \rangle$, etc.

Exercise 10: Write a SINGULAR procedure `spolynomial` which takes as input two polynomials and returns their s-polynomial.

Exercise 11: Write a SINGULAR procedure `NFBuchberger` which takes as input a list consisting of a polynomial g and a list of polynomials $f_1, \ldots, f_k$ and returns a normal form of g with respect to $(f_1, \ldots, f_k)$.

Note, the ordering of the base ring must be global! For a list of polynomials it is best to use the type `ideal`.

Exercise 12: Write a SINGULAR procedure `RedNFBuchberger` which takes as input a list consisting of a polynomial g and a list of polynomials $f_1, \ldots, f_k$ and returns the reduced normal form of g with respect to $(f_1, \ldots, f_k)$.

Note, the ordering of the base ring must be global!