Fachbereich Mathematik
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Summer Semester 2004, Set 4

Computer Algebra

Due date: Monday, 07/06/2004, 10h00

Exercise 13: Apply NFBUCHBERGER to the following data (without using `Singular`):

$$g = x^4 + y^4 + z^4 + xyz, G = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right), >=>_{dp}.$$

Exercise 14: Let $>$ be any monomial ordering on Mon_n, and let $f, g \in K[\underline{x}]$ with $\gcd\big(\mathrm{lm}(f), \mathrm{lm}(g)\big) = 1$. Show there is a polynomial normal form PNF on $K[\underline{x}]_>$ such that

$$\mathrm{PNF}\big(\mathrm{spoly}(f,g), (f,g)\big) = 0.$$

Remark, this is the so-called product criterion for the spoly-computation, which allows to "forget" many spoly's in the standard basis computation! By Buchberger's Criterion 4.9 this implies that (f, g) is a standard basis and that any polynomial normal form PNF has this property.

Hint, show first that $\mathrm{spoly}(f, g) = a_0 f + b_0 g$ for $a_0 = -\mathrm{tail}(g)$ and $b_0 = \mathrm{tail}(f)$, and then define recursively $a_i = \mathrm{tail}(a_{i-1})$ and $b_i = \mathrm{tail}(b_{i-1})$. Consider the maximal value N such that $u \cdot \mathrm{spoly}(f, g) = a_N f + b_N g$ for some unit $u \in K[\underline{x}]_>^* \cap K[\underline{x}]$, and distinguish the two cases that $\mathrm{lt}(a_N f) + \mathrm{lt}(b_N g)$ vanishes respectively does not vanish.

Exercise 15: Write a SINGULAR procedure `PNFMora` which takes as input a list consisting of a polynomial g and a list of polynomials $f_1, \ldots, f_k$ and returns a polynomial normal form of g with respect to $(f_1, \ldots, f_k)$.

Exercise 16: Deduce from the proof of Mora's polynomial normal form algorithm a recursive algorithm which takes as input a ploynomial $g \in K[\underline{x}]$ and a list of polynomials $G = (f_1, \ldots, f_k)$ and returns a list containing the following data: a unit $u \in K[\underline{x}]_>^* \cap K[\underline{x}]$, a list of polynomials $q_1, \ldots, q_k$, and a polynomial r, such that $ug = \sum_{i=1}^k q_i f_i + r$ is a standard representation. Then implement this algorithm as procedure `PDwR` in SINGULAR.

Note that the correctness and the termination follows from the termination and correctness of PNFMora, since it is just an extension of the algorithm which returns the unit u and the q_i as well, so that you just have to formulate and to implement the algorithm!