Some Remarks on the Graded Lemma of Nakayama

Let $R = \bigoplus_{d \geq 0} R_d$ be a *positively graded* ring with R_0 a *field*, and let $M = \sum_{d \geq d_0} M_d$ and $N = \sum_{d \geq e_0} N_d$ be *finitely generated* graded modules. By $\mathfrak{m} = R_{>0} = \bigoplus_{d>0} R_d$ we denote the homogeneous maximal ideal of elements of positive degree.

Whenever we consider morphisms $\varphi : M \to N$ between graded modules, they will be graded of degree zero, i. e. $\varphi(M_d) \subseteq N_d$, and we thus have the restriction $\varphi_d : M_d \to N_d$ which is just a morphism of R_0-vector spaces.

Lemma 1 (Nakayama 1)
Let $I \subseteq \mathfrak{m}$ be a homogeneous ideal such that $I \cdot M = M$, then $M = 0$.

Proof: Let $I = \bigoplus_{d \geq d_1} I_d$ with $d_1 > 0$ and suppose that $M \neq 0$. We may thus assume that $M_{d_0} \neq 0$. But then the minimal degree of an element in $I \cdot M$ will be $d_0 + d_1 > d_0$ in contradiction to $M_{d_0} \subset I \cdot M$. □

Corollary 2 (Nakayama 2)
Let $N \subseteq M$ be a graded submodule of M such that $M \subseteq N + \mathfrak{m} \cdot M$, then $N = M$.

Proof: By assumption we have

$$\mathfrak{m} \cdot (M/N) \subseteq (\mathfrak{m} \cdot M + N)/N = M/N,$$

so that by Nakayama 1 $M/N = 0$. □

Corollary 3 (Nakayama 3)
Let $m_1, \ldots, m_r \in M$ be homogeneous elements such that $\overline{m_1}, \ldots, \overline{m_r}$ are a basis of the R_0-vector space $M/\mathfrak{m} \cdot M$, then $m_1, \ldots, m_r$ are a minimal generating system of M.

Proof: Let $N = \langle m_1, \ldots, m_r \rangle$, then by assumption N is a graded submodule of M such that $M = N + \mathfrak{m} \cdot M$. So by Nakayama $N = M$, i. e. $m_1, \ldots, m_r$ is a generating system of M. Moreover, if there was a generating system with fewer elements, their residue classes would generate $M/\mathfrak{m} \cdot M$, in contradiction that this is a vector space of dimension r. □

Corollary 4 (Nakayama 4)
If $\varphi : M \to N$ is a graded morphism of degree zero such that[1] $\overline{\varphi} : M/\mathfrak{m} \cdot M \to N/\mathfrak{m} \cdot N$ is an epimorphism, then φ is an epimorphism.

Proof: By assumption we have

$$0 = (N/\mathfrak{m} \cdot N)/\operatorname{Im}(\overline{\varphi}) = (N/\mathfrak{m} \cdot N)/(\operatorname{Im}(\varphi) + \mathfrak{m} \cdot N/ \cdot N) \cong N/(\operatorname{Im}(\varphi) + \mathfrak{m} \cdot N).$$

Thus $N = \operatorname{Im}(\varphi) + \mathfrak{m} \cdot N$ and by Nakayama 2 we have $N = \operatorname{Im}(\varphi)$, i. e. φ is surjective. □

[1] Note, that $\overline{\varphi}$ is actually just φ_0.

Lemma 5
If $\varphi : M \longrightarrow M$ is a graded morphism of degree zero which is surjective, then φ is an isomorphism.[2]

Proof: By assumption $\varphi_d : M_d \longrightarrow M_d$ is an epimorphism of finite-dimensional vector spaces of the same dimension, hence it is an isomorphism for all $d \geq d_0$. But then φ is an isomorphism. □

Corollary 6
If M is projective, then M is free.

Proof: By Nakayama 3 we may choose a minimal system of homogeneous generators $m_1, \ldots, m_r$ of M which project to a basis of $M/\mathfrak{m} \cdot M$. Let $a_i = -\deg(m_i)$, $L = \bigoplus_{i=1}^r R(a_i)$ and consider

$$\varphi : L \longrightarrow M : e_i \mapsto m_i.$$

φ is a surjective graded morphism of degree 0, and since M is projective this morphism splits, in particular we may assume

$$L = M \bigoplus \ker(\varphi).$$

However,

$$\overline{\varphi} : L/\mathfrak{m} \cdot L \cong R_0^r \longrightarrow M/\mathfrak{m} \cdot M \cong R_0^r : e_i \mapsto \overline{m_i}$$

maps a basis to a basis and is thus an isomorphism of vector spaces. Thus $\ker(\varphi) + \mathcal{M} \cdot L/\mathfrak{m} \cdot L = \ker\left(\overline{\varphi}\right) = 0$, i. e. $\ker(\varphi) \subseteq \mathfrak{m} \cdot L$.
This implies

$$L = \ker(\varphi) + M \subseteq \mathfrak{m} \cdot L + M,$$

and Nakayama 2 implies that $L = M$. □

Alternative Proof: As above we may choose a minimal set of generators $m_1, \ldots, m_r$ and define the epimorphism $\varphi : L \longrightarrow M$. Since M is projective, φ splits, i. e. there is a morphism $\psi : M \longrightarrow L$ such that $\varphi \circ \psi = \mathrm{id}_M$. We want to show, that $\psi \circ \varphi$ is an isomorphism, which then implies that φ is a monomorphism as well.
By assumption $\overline{\psi \circ \varphi} = \overline{\psi} \circ \overline{\varphi} : L/\mathfrak{m} \cdot L \longrightarrow L/\mathfrak{m} \cdot L$ is an isomorphism of vectorspaces and hence by Nakayama 4 $\psi \circ \varphi$ is an epimorphism. But then by the above Lemma $\psi \circ \varphi$ is an isomorphism. □

Let us conclude this note by an application of the Lemma of Nakayama for local rings in order to show that certain graded rings are isomorphic.

Theorem 7
Let $(S, \mathfrak{n})$ be a local *noetherian ring of dimension d. Then R is regular if and only if*

$$S/\mathfrak{n}[t_1, \ldots, t_d] \cong \mathrm{Gr}_{\mathfrak{n}}(S) := \bigoplus_{k \geq 0} \mathfrak{n}^k / \mathfrak{n}^{k+1}$$

as graded $S/\mathfrak{n}$-algebras.

[2] See [Eis96] Corollary 4.4 for a non-graded version!

Proof: Let us first assume that $\varphi : S/\mathfrak{n}[t_1,\ldots,t_d] \to \mathrm{Gr}_{\mathfrak{n}}(S)$ is an isomorphism, and let $G_{>0} = \bigoplus_{k\geq 1} \mathfrak{n}^k/\mathfrak{n}^{k+1}$ be the unique homogeneous maximal ideal of $\mathrm{Gr}_{\mathfrak{n}}(S)$. Then φ induces the $S/\mathfrak{n}$-vector space isomorphism

$$\langle t_1,\ldots,t_d\rangle/\langle t_1,\ldots,t_d\rangle^2 \;\cong\; G_{>0}/G_{>0}^2 \;\cong \mathfrak{n}/\mathfrak{n}^2.$$

Thus the vector space $\mathfrak{n}/\mathfrak{n}^2$ can generated by d elements, and we have

$$\dim(S) = d \geq \dim_{S/\mathfrak{n}}(\mathfrak{n}/\mathfrak{n}^2) \geq \dim(S).$$

This implies $\dim(S) = \dim_{S/\mathfrak{n}}(\mathfrak{n}/\mathfrak{n}^2)$, and thus S is regular.
We prove the converse only for the case that $S/\mathfrak{n}$ is an infinite field. Let us therefore now assume that S is regular and $\#S/\mathfrak{n} = \infty$.
If S is regular of dimension d then the maximal ideal is generated by a system of parameters

$$\mathfrak{n} = \langle x_1,\ldots,x_d\rangle.$$

Consider therefore the graded $S/\mathfrak{n}$-algebraepimorphism defined by

$$\varphi : S/\mathfrak{n}[t_1,\ldots,t_d] \longrightarrow \mathrm{Gr}_{\mathfrak{n}}(S) : t_i \mapsto \overline{x_i} = x_i + \mathfrak{n}^2,$$

and let $I = \ker(\varphi)$ be its kernel, which is a graded ideal in the polynomial ring $S/\mathfrak{n}[t_1,\ldots,t_d]$. By the Theorem of Noether-Normalisation and since $S/\mathfrak{n}$ is infinite, there is a finite $S/\mathfrak{n}$-algebrahomomorphism

$$S/\mathfrak{n}[\overline{y_1},\ldots,\overline{y_e}] \hookrightarrow (S/\mathfrak{n}[t_1,\ldots,t_d])/I, \tag{1}$$

where $e = \dim\big((S/\mathfrak{n}[t_1,\ldots,t_d])/I\big)$ and where the y_i are suitable $S/\mathfrak{n}$-linear combinations of the t_j. In particular, the y_i are homogeneous of degree one.
Suppose now that $I \neq 0$, i.e. the φ is not an isomorphism. Then $e < d$. Note that $\varphi(y_i) \in \mathfrak{n}/\mathfrak{n}^2$, and therefore there exists a $z_i \in \mathfrak{n}$ such that

$$z_i \equiv \varphi(y_i) \pmod{\mathfrak{n}^2}.$$

Let us consider the ideal

$$J = \langle z_1,\ldots,z_e\rangle_S \subset \mathfrak{n}$$

in the ring S. Since (1) is finite, t_i is integral over $S/\mathfrak{n}\big[\overline{y_1},\ldots,\overline{y_e}\big]$, and thus there exist equations

$$t_i^{n_i} + \sum_{j=0}^{n_i-1} a_{i,j}\cdot t_i^j \in I,$$

with $a_{i,j} \in S/\mathfrak{n}[y_1,\ldots,y_e]$. Multiplying the term with $t_i^{n-n_i}$, $n = \max\{n_j \mid j = 1,\ldots,d\}$, we may assume that $n_i = n$ for all $i = 1,\ldots,d$. And considering only the homogeneous part of degree n of the term we may assume that $a_{i,j}$ is homogeneous of degree $n-j \geq 1$ for all i,j. If we now apply φ we get the equation

$$\overline{x_i}^n = \varphi(t_i^n) = -\sum_{j=0}^{n-1} \varphi(a_{i,j})\cdot \overline{x_i}^j,$$

where

$$\varphi(a_{i,j}) \in \big\langle \varphi(y_1),\ldots,\varphi(y_e)\big\rangle^{n-j}.$$

But this implies that

$$x_i^n \in J + \mathfrak{n}^{n+1},$$

and thus

$$\mathfrak{n}^{dn} = \langle x_1^n, \ldots, x_d^n \rangle \cdot \mathfrak{n}^{dn-n} \subseteq (J + \mathfrak{n}^{n+1}) \cdot \mathfrak{n}^{dn-n} \subseteq J + \mathfrak{n}^{dn+1}.$$

By Nakayama 2 for local rings this leads to

$$\mathfrak{n}^{dn} \subseteq J,$$

and since $J \subseteq \mathfrak{n}$ this implies that J is an $\mathfrak{n}$-primary ideal generated by e elements. This, however, is in contradiction to the fact that $\dim(S) = d > e$, since the dimension of a local ring is the minimal number of generators of an $\mathfrak{n}$-primary ideal (see [Eis96], Corollary 10.7). □

Remark 8

If the field $S/\mathfrak{n}$ in the previous theorem is not infinite, then a suitable version of the Theorem of Noether-Normalisation (see [ZS60] Chap. VII, Theorem 25, p. 200) shows that one may still gets a finite extension

$$S/\mathfrak{n}[\overline{y_1}, \ldots, \overline{y_e}] \hookrightarrow (S/\mathfrak{n}[t_1, \ldots, t_d])/I,$$

where the y_i are homogeneous – not necessarily of degree one. One may even assume that they are of the same degree, say m, just replacing the y_i by suitable powers. Then the z_i in the above proof can be chosen in $\mathfrak{n}^m$ such that

$$z_i \equiv \varphi(y_i) \pmod{\mathfrak{n}^{m+1}},$$

and the proof generalises in the obvious way.

References

[Eis96] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, no. 150, Springer, 1996.

[ZS60] Oscar Zariski and Pierre Samuel, *Commutative algebra*, vol. II, Graduate Texts in Mathematics, no. 29, Springer, 1960.