Computer Algebra

Exercise 1: Let $I = \left\langle \underline{x}^\alpha \cdot e_i \mid (\alpha, i) \in \Lambda \right\rangle$ and $J = \left\langle \underline{x}^\beta \cdot e_j \mid (\beta, j) \in \Lambda' \right\rangle$ be two monomial submodules of $R[\underline{x}]^m$ and let $\underline{x}^\gamma \cdot e_k \in R[\underline{x}]^m$ be a monomial. Show that

a. $I \cap J = \left\langle \mathrm{lcm}\left(\underline{x}^\alpha \cdot e_i, \underline{x}^\beta \cdot e_j\right) \mid (\alpha, i) \in \Lambda, (\beta, j) \in \Lambda' \right\rangle$, and

b. $I : \langle \underline{x}^\gamma \cdot e_k \rangle = \left\langle \frac{\mathrm{lcm}\left(\underline{x}^\alpha, \underline{x}^\gamma\right)}{\underline{x}^\gamma} \;\middle|\; (\alpha, k) \in \Lambda \right\rangle$.

Exercise 2: Let $>$ be any monomial ordering on Mon_n.

a. Show that for a fixed $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ the following defines a monomial ordering on Mon_n:

$$\underline{x}^\alpha >_{(w,>)} \underline{x}^\beta \;:\Longleftrightarrow\; \langle w, \alpha \rangle > \langle w, \beta \rangle, \;\text{ or }\; \big(\langle w, \alpha \rangle = \langle w, \beta \rangle \;\text{ and }\; \underline{x}^\alpha > \underline{x}^\beta\big).$$

Under which assumptions is the above definition independent of the chosen ordering $>$?

b. Let $A \in \mathrm{Gl}_n(\mathbb{Q})$ be an invertible $n \times n$-matrix over the rational numbers. Show that the following defines a monomial ordering on Mon_n:

$$\underline{x}^\alpha >_{(A,>)} \underline{x}^\beta \;:\Longleftrightarrow\; \underline{x}^{A\alpha} > \underline{x}^{A\beta}.$$

Exercise 3: Determine matrices $A \in \mathrm{Gl}_n(\mathbb{R})$ which define the following orderings:

a. $(>_{lp}, >_{ds})$, here $n = n_1 + n_2$ with n_1 variables for lp and n_2 for ds;

b. $(>_{dp}, >_{ls})$, here $n = n_1 + n_2$ with n_1 variables for dp and n_2 for ls;

c. $>_{wp(5,3,4)}$, here $n = 3$;

d. $>_{ws(5,3,4)}$, here $n = 3$.

Exercise 4: We define the *degree lexicographical ordering* $>_{Dp}$ on Mon_n by

$$\underline{x}^\alpha >_{Wp} \underline{x}^\beta \;\Longleftrightarrow\; |\alpha| > |\beta| \text{ or } \big(|\alpha| = |\beta| \text{ and } \exists\, k : \alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k\big).$$

Show that the orderings $>_{lp}$, $>_{Dp}$ and $>_{dp}$ are described by the following characterising properties. Let $>$ be a monomial ordering on Mon_n, then:

a. $> = >_{lp}$ if and only if $>$ is an elimination ordering for $\{x_1, \ldots, x_k\}$ for all $k = 1, \ldots, n-1$, i. e. if $\mathrm{lm}(f) \in R[x_{k+1}, \ldots, x_n]$ implies $f \in R[x_{k+1}, \ldots, x_n]$.

b. $> = >_{Dp}$ if and only if $>$ is a degree ordering and for any *homogeneous* $f \in R[\underline{x}]$ with $\mathrm{lm}(f) \in R[x_k, \ldots, x_n]$ we have $f \in R[x_k, \ldots, x_n]$, $k = 1, \ldots, n$.

c. $> = >_{dp}$ if and only if $>$ is a degree ordering and for any *homogeneous* $f \in R[\underline{x}]$ with $\mathrm{lm}(f) \in \langle x_k, \ldots, x_n \rangle$ we have $f \in \langle x_k, \ldots, x_n \rangle$, $k = 1, \ldots, n$.