Fachbereich Mathematik
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Computer Algebra

Exercise 25: Let $>$ be a global monomial ordering on $\mathrm{Mon}(\underline{x})$, and let $M \in \mathrm{Mat}(n \times n, K[\underline{x}])$. By $f_1, \ldots, f_n \in K[\underline{x}]^{2n}$ we denote the rows of the matrix $(M, \mathbb{1}_n)$, and $G = (g_1, \ldots, g_k)$ shall be *the* reduced standard basis of $\langle f_1, \ldots, f_k \rangle_{K[\underline{x}]} \leq K[\underline{x}]^{2n}$ w. r. t. the ordering $(c, >)$ with $\mathrm{lm}(g_1) > \ldots > \mathrm{lm}(g_k)$. Show:

a. M is invertible if and only if $k = n$ and $\mathrm{lm}(g_i) = e_i$ for $i = 1, \ldots, n$.

b. If M is invertible, then the rows of $(\mathbb{1}_n, M^{-1})$ are just $g_1, \ldots, g_n$.

Exercise 26: Let $f, g \in K[\underline{x}]$. Express $\gcd(f, g)$ and $\mathrm{lcm}(f, g)$ in terms of elements in $\mathrm{syz}(f, g)$ and derive an algorithm to compute these, assuming we can compute a standard basis of $\mathrm{syz}(f, g)$.

Exercise 27: Let $R = \mathbb{Q}[x, y, z]/\langle x^2+y^2+z^2 \rangle$, $M = R^3/\langle (x, xy, xz)^t \rangle$ and $N = R^2/\langle (1, y)^t \rangle$. Moreover, let $\varphi : M \to N$ be given by the matrix

$$A = \begin{pmatrix} x^2+1 & y & z \\ yz & 1 & -y \end{pmatrix}.$$

a. Compute $\mathrm{Ker}(\varphi)$.

b. Test if $(x^2, y^2)^t \in \mathrm{Im}(\varphi)$.

c. Compute $\mathrm{Im}(\varphi) \cap \{f \in N \mid f \equiv (h, 0) \bmod \langle (x, 1)^t \rangle \text{ for some } h \in R\}$.

d. Compute $\mathrm{ann}_R\big(\mathrm{Im}(\varphi)\big)$.

Note, you may use Singular for your computations!

Exercise 28: Write a SINGULAR procedure `chaincriterion` which takes a list of pairs of polynomials as input and eliminates pairs using the chain criterion. Then adjust your procedure `standardbasis` with this procedure.