Fachbereich Mathematik — Summer Semester 2011, Set 4

Thomas Markwig

Commutative Algebra

Exercise 12: Let M be an R-module.

a. Prove that $\mu : M \to \operatorname{Hom}_R(R, M)$ with $\mu(m) : R \to M : r \mapsto r \cdot m$ is an isomorphism.

b. Give an example where $M \not\cong \operatorname{Hom}_R(M, R)$.

Exercise 13: Let R be an integral domain and $0 \neq I \trianglelefteq R$.
Show that I as R-module is free if and only if I is principal.

Exercise 14: Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers. Consider the R-linear map $\varphi : R^3 \to R^2 : m \mapsto A \cdot m$ where

$$A = \begin{pmatrix} 1 + x^4 - x^7 + 3x^{100} & \cos(x) & 2 - \exp(x) \\ x^4 - 5x^8 & \sum_{i=0}^{\infty}(5x + x^2)^i & 0 \end{pmatrix} \in \operatorname{Mat}(2 \times 3, R).$$

Is φ an epimorphism?

Exercise 15: Let $p \in \mathbb{Z}$ be a prime number. Consider the subring $R = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, p \nmid b \right\} \leq \mathbb{Q}$ of the rational numbers, and consider $M = \mathbb{Q}$ as an R-module.

a. Show that R is local with maximal ideal $\mathfrak{m} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, p \nmid b, p \mid a \right\}$.

b. $\mathfrak{m} \cdot M = M$, but $M \neq 0$.

c. Find a set of generators for M.

In-Class Exercise 10: Consider $R = \mathbb{R}[x, y, z]$ and $M = \langle xy, xz, yz \rangle$. Find a polynomial $F \in R[t]$ such that $F(\varphi) = 0$ where is the restriction to M of the map

$$R \longrightarrow R : f \mapsto f \cdot (x + y + z).$$

In-Class Exercise 11: What is the K-vector space dimension of the cokernel of the $K[x]$-linear map $\varphi : K[x]^2 \longrightarrow K[x]^2 : (a, b) \mapsto (a + b, x^2 \cdot b)$?