

## Commutative Algebra

**Exercise 28:** Let  $M$  be an  $R$ -module and  $\varphi : M \rightarrow M$  an  $R$ -linear map. Show:

- If  $M$  is noetherian and  $\varphi$  is surjective, then  $\varphi$  is an isomorphism.
- If  $M$  is artinian and  $\varphi$  is injective, then  $\varphi$  is an isomorphism.

Hint, consider the kernel respectively cokernel of  $\varphi^n$  for  $n \in \mathbb{N}$ .

**Exercise 29:** Which of the following rings  $R_i$  is noetherian?

- $R_1 = \left\{ \frac{g}{h} \in \text{Quot}(\mathbb{C}[x]) \mid h(z) \neq 0 \text{ for } |z| = 1 \right\}$ .
- $R_2 = \{f \in \mathbb{C}\{x\} \mid f \text{ has infinite radius of convergence}\}$ .
- $R_3 = \{f \in \mathbb{C}[x] \mid \frac{\partial^i f}{\partial x^i}(0) = 0 \text{ for } i = 1, \dots, k\}$ ,  $k$  fixed.

**Exercise 30:** Let  $\mathbb{Q} \subseteq K$  be a field extension such that  $\dim_{\mathbb{Q}}(K) < \infty$ , and suppose  $R$  is a subring of  $K$  containing  $\mathbb{Z}$  such that  $I \cap \mathbb{Z} \neq \{0\}$  for each ideal  $0 \neq I \leq R$ . Show that  $R$  is noetherian.

Hint, show first that  $\dim_{\mathbb{Z}/p\mathbb{Z}}(R/pR) \leq \dim_{\mathbb{Q}}(K)$  for any prime number  $p$ . Then conclude that for  $0 \neq m \in I \cap \mathbb{Z}$  the set  $R/mR$  (and hence  $I/mR$ ) is finite by induction on the number of prime factors of  $m = p_1 \cdots p_k$ ,  $p_i$  prime number. – Remark: using a bit field theory one can show that the assumption  $I \cap \mathbb{Z} \neq \{0\}$  is always fulfilled.

**Exercise 31:** Let  $R \subseteq R' \subseteq R''$  be rings,  $R'' = R[a_1, \dots, a_n]$  a finitely generated  $R$ -algebra and  $R''$  finitely generated as an  $R'$ -module. Show, if  $R$  noetherian, then  $R'$  is finitely generated as an  $R$ -algebra and noetherian.

Recall,  $R[a_1, \dots, a_n] = \{f(a_1, \dots, a_n) \mid f \in R[x_1, \dots, x_n]\}$  is the set of all polynomial expressions in  $a_1, \dots, a_n$  with coefficients in  $R$ . Hint, if  $R'' = \langle b_1, \dots, b_m \rangle_{R'}$ , then write  $a_i$  and  $b_i \cdot b_j$  as linear combinations of the  $b_v$  and consider the  $R$ -algebra generated by the coefficients.

**In-Class Exercise 17:** Show that  $\mathbb{C}[x, y]/I$  with  $I = \langle x^3 - x^2, x^2y + 2x^2, xy - y, y^2 + 2y \rangle$  is an artinian ring and decompose it as a direct sum of two local artinian rings.

**In-Class Exercise 18:** Is  $K(x) = \text{Quot}(K[x])$  a noetherian  $K[x]_{\langle x \rangle}$ -module?