

Commutative Algebra

Exercise 42 a., b. and c. need NOT be handed in for marking.

Exercise 40: Let $R \subset R'$ be an integral ring extension and let $m' \triangleleft R'$ be a maximal ideal such that $m = m' \cap R \triangleleft R$ is maximal as well. Is then R'_m integral over R_m ?

Hint, consider $R = K[x^2 - 1]$, $R' = K[x]$, $m' = \langle x - 1 \rangle$, and $f = \frac{1}{1+x} \in R'_m$.

Exercise 41: Let $R \subset R'$ be integral domains and $f, g \in R'[x]$ be monic polynomials. Show that if $f \cdot g \in \text{Int}_{R'}(R)[x]$, then $f, g \in \text{Int}_{R'}(R)[x]$.

Note, if we apply this to $R = \mathbb{Z}$ and $R' = \mathbb{Q}$, then we get for monic $f, g \in \mathbb{Q}[x]$ that $f \cdot g \in \mathbb{Z}[x]$ implies $f, g \in \mathbb{Z}[x]$.

Exercise 42: [Rings of Integers of Quadratic Number Fields]

Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree number (i.e. no square a^2 divides d), then $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$ is a field extension of \mathbb{Q} with $\dim_{\mathbb{Q}} \mathbb{Q}[\sqrt{d}] = 2$. Consider the conjugation

$$C : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q}[\sqrt{d}] : a + b\sqrt{d} \mapsto a - b\sqrt{d},$$

the norm

$$N : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) \cdot C(a + b\sqrt{d}) = a^2 - b^2d,$$

and the trace

$$T : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) + C(a + b\sqrt{d}) = 2a.$$

Show:

- $C(x \cdot y) = C(x) \cdot C(y)$ and $N(x \cdot y) = N(x) \cdot N(y)$ for $x, y \in \mathbb{Q}[\sqrt{d}]$.
- C and T are \mathbb{Q} -linear.
- If $x \in \mathbb{Q}[\sqrt{d}] \setminus \mathbb{Q}$, then $\mu_x = (t - x) \cdot (t - C(x)) = t^2 - T(x) \cdot t + N(x) \in \mathbb{Q}[t]$ is the minimal polynomial of x over \mathbb{Q} .
- $x \in \mathbb{Q}[\sqrt{d}]$ is integral over \mathbb{Z} if and only if $T(x)$ and $N(x)$ are integers.
- $\text{Int}_{\mathbb{Q}[\sqrt{d}]}(\mathbb{Z}) = \mathbb{Z}[\omega_d]$, where $\omega_d = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod{4} \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$

In-Class Exercise 23: Which of the following ring extensions is integral?

- $K[x] \hookrightarrow K[x, y, z] / \langle z^2 - xy \rangle : x \mapsto \bar{x}$.
- $K[x] \hookrightarrow K[x, y, z] / \langle z^2 - xy, y^3 - x^2 \rangle : x \mapsto \bar{x}$.
- $K[x] \hookrightarrow K[x, y, z] / \langle z^2 - xy, x^3 - yz \rangle : x \mapsto \bar{x}$.