Fachbereich Mathematik
Thomas Markwig

Summer Semester 2011, Set 12

Commutative Algebra

Exercise 43:

a. Show that every maximal ideal in $\mathbb{Z}[x]$ can generated by a prime number $p \in \mathbb{Z}$ and a polynomial $f \in \mathbb{Z}[x]$ such that its residue class in $\mathbb{Z}_p[x]$ is irreducible.

b. Show that $\dim(\mathbb{Z}[x]) = 2$.

Hint, show in a. first that a maximal ideal cannot be principal, then that it contains a prime number and finally that modulo that prime number it will be generated by a single polynomial. Recall that $\mathbb{Z}[x]/\langle p\rangle \cong \mathbb{Z}_p[x]$ is a PID and that in $\mathbb{Q}[x]$ the Bézout identity holds.

Exercise 44: Let $K \subseteq K'$ be a *field* extension, and let $T \subset K'$ (possibly infinite). T is called *algebraically independent* over K if every finite subset of T is algebraically independent over K. And an algebraically independent set T is a *transcendence basis* of K'/K if $T \cup \{t'\}$ is algebraically dependent for every $t' \in K' \setminus T$. Show:

a. An algebraically independent set T is a transcendence basis of K'/K if and only K' is integral over

$$K(T) = \left\{ \frac{f(t_1, \ldots, t_n)}{g(t_1, \ldots, t_n)} \mid f, g \in K[x_1, \ldots, x_n], g \neq 0, t_1, \ldots, t_n \in T, n \geq 1 \right\}.$$

b. If T and T' are transcendence bases of K'/K and $t \in T$, then there is a $t' \in T'$ such that $(T \setminus \{t\}) \cup \{t'\}$ is a transcendence basis of K'/K.

c. If T and T' are transcendence bases of K'/K, $|T| < \infty$, then $\mathrm{trdeg}_K(K') = |T| = |T'|$.

d. $\mathrm{trdeg}_K \big(K(x_1, \ldots, x_n)\big) = n$.

Hint for part b., if $T_0 = T \setminus \{t\}$, then consider the field extensions $K(T_0) \subset K'$, $K(T' \cup T_0) \subset K'$ and $K(T_0) \subset K(T' \cup T_0) = K(T_0)(T')$. Which of these are integral (which is the same as algebraic)?

Exercise 45: Find a Noether normalisation of $R = K[x, y]/\langle x^3 - y^2\rangle$ and compute the normalisation of R.

Exercise 46: Let R be a finitely generated K-algebra which is an integral domain and let $K' = \mathrm{Quot}(R)$. Show that:

a. If $\beta_1, \ldots, \beta_d \in R$ are algebraically independent over K and R is algebraic over $K[\beta_1, \ldots, \beta_d]$, then $\mathrm{Quot}(R)$ is algebraic over $K(\beta_1, \ldots, \beta_d)$.

b. $\mathrm{trdeg}_K(R) = \mathrm{trdeg}_K(K')$.

In-Class Exercise 24: Find all maximal ideals in $\mathbb{C}[x, y]/\langle x^3 - x^2, x^2y - 2x^2\rangle$.

In-Class Exercise 25: Compute $\mathrm{Quot}(R)$ and $\dim(\mathrm{Quot}(R))$ for $R = K[x, y]/\langle x^2, xy\rangle$.