Fachbereich Mathematik
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Commutative Algebra

Exercise 47: Prove the Algebraic HNS (Theorem 7.1) using Noether-Normalisation.

Exercise 48: Let R be a ring. Show that $\dim(R[x]) \geq \dim(R) + 1$.

Hint, consider ideals of the form $I[x] = \left\{ \sum_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in I \right\}$ for some ideal $I \trianglelefteq R$. – Note, if R is noetherian one can actually show equality, but that is much harder.

Exercise 49: Let R be an integral domain. Show:

a. R is a valuation ring if and only if for two ideals $I, J \trianglelefteq R$ we have $I \subseteq J$ or $J \subseteq I$.

b. If R is a valuation ring and $P \in \operatorname{Spec}(R)$, then R_P and R/P are valuation rings.

Exercise 50: **[A valuation on the field $K\{\{t\}\}$]**

Let $K\{\{t\}\}$ be the field from Exercise 3.

a. Show that $\operatorname{ord} : (K\{\{t\}\}^*, *) \to (\mathbb{R}, +) : f \mapsto \min\{\alpha \in \mathbb{R} \mid f(\alpha) \neq 0\}$ is a valuation.

b. R_{ord} is not noetherian, hence ord is not discrete, but $\dim(R_{\operatorname{ord}}) = 1$.

c. If $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ are algebraically independent over $\mathbb{Q}$, then $(t^{\alpha_1}, \ldots, t^{\alpha_n})$ are algebraically independent over K. In particular, $\operatorname{trdeg}_K \left(K\{\{t\}\}\right) = \infty$.

Hint for part b., note that $\mathfrak{m}_{R_{\operatorname{ord}}} = \langle t^\alpha \mid \alpha > 0 \rangle$, where $t^\alpha : \mathbb{R} \to \mathbb{R}$ satisfies $t^\alpha(\alpha) = 1$ and $t^\alpha(\beta) = 0$ for $\beta \neq \alpha$.

In-Class Exercise 26: Compute the dimension of $K[x, y, z]_{\langle x^2 - yz \rangle}$.

In-Class Exercise 27: Is the ring $\mathbb{C}[x, y]/\langle x^2 - y^2 - y^3 \rangle$ a Dedekind domain?

In-Class Exercise 28: Is the ring $\mathbb{C}[x, y]/\langle x - y^2 - y^3 \rangle$ a Dedekind domain?