Fachbereich Mathematik Summer Semester 2011, Set 14
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Commutative Algebra

Exercise 51: Let K be any field, and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ be an independent set of real numbers. Show:

a. $\varphi_{\underline{\alpha}} : K(x_1 \ldots, x_n) \to K\{\{t\}\} : \frac{f}{g} \mapsto \frac{f(t^{\alpha_1}, \ldots, t^{\alpha_n})}{g(t^{\alpha_1}, \ldots, t^{\alpha_n})}$ is a K-algebra*mono*morphism.

b. $\nu : K(x_1, \ldots, x_n)^* \mapsto \mathbb{R} : h \mapsto (\mathrm{ord} \circ \varphi_{\underline{\alpha}})(h)$ is a valuation of $K(x_1, \ldots, x_n)$.

c. $1 = \dim(R_\nu) < \mathrm{trdeg}_K \left(K(x_1, \ldots, x_n)\right) - \mathrm{trdeg}_K \left(R_\nu / \mathfrak{m}_{R_\nu}\right) = n$, for $n \geq 2$.

Note, $\mathrm{ord} : K\{\{t\}\}^* \to \mathbb{R}$ is the valuation of $K\{\{t\}\}$ from Exercise 50.

Exercise 52: Let R be a Dedekind domain and $0 \notin S \subset R$ multiplicatively closed. Show that either $S^{-1}R = \mathrm{Quot}(R)$ or $S^{-1}R$ is a Dedekind domain.

Exercise 53: [Lemma of Gauß]*
Let R be a Dedekind domain. For a polynomial $f = \sum_{i=0}^n a_i x^i \in R[x]$ we call $c(f) = \langle a_0, \ldots, a_n \rangle_R$ the *content* of f. Show that $c(f) \cdot c(g) = c(f \cdot g)$ for $f, g \in R[x]$.

Hint, reduce to the case that R is local (i.e. a DVR), and use Nakayama's Lemma in a suitable way.

Exercise 54: [Chinese Remainder Theorem]
Let R be a Dedekind domain and $I_1, \ldots, I_n \trianglelefteq R$.

a. Show that the following sequence is exact

$$R \xrightarrow{\varphi} \bigoplus_{i=1}^{n} R/I_i \xrightarrow{\psi} \bigoplus_{i<j} R/(I_i + I_j),$$

where $\varphi(x) = (x + I_1, \ldots, x + I_n)$ and $\psi(x_1 + I_1, \ldots, x_n + I_n) = (x_i - x_j + I_i + I_j)_{i<j}$.

b. Given $x_1, \ldots, x_n \in R$. Show there is an $x \in R$ such that $x \equiv x_i \pmod{I_i}$ for $i = 1, \ldots, n$ if and only if $x_i \equiv x_j \pmod{I_i + I_j}$ for $i \neq j$.

Hint for part a., localize with respect to maximal ideals! – Note, part b. generalizes 1.12.

In-Class Exercise 29: Describe $\mathrm{Div}(R)$ and $\mathrm{Pic}(R)$ for $R = K[x,y]/\langle y - x^2 \rangle$?

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*What is the connection to the *Lemma of Gauß* in 1.38, stating "R factorial implies $R[x]$ factorial"? If we replace the assumption "R Dedekind domain" by "R UFD" the above result holds true as well. Call a polynomial *primitive* if $c(f) = R$ (or equivalently if R^* is the gcd of the coefficients of f), then we deduce from the above result that a primitive polynomial in $R[x]$ can only factorize in a product of primitive polynomials, which are then necessarily of smaller degree. By induction on the degree we see that each primitive polynomial is a product of irreducible primitive polynomials. Thus, every polynomial is a product of irreducible ones, since splitting off a greatest common divisor g of its coefficients gives a primitive one and g factorises since R is factorial. – It then only remains to show that each irreducible polynomial in $R[x]$ is prime. – In the literature it is more common to call the statement "R UFD implies $c(f \cdot g) = c(f) \cdot c(g)$" the *Lemma of Gauß*.