Fachbereich Mathematik
Dr. Thomas Keilen

Winter Semester 2004/05, Set 1

Commutative Algebra

Due date: Monday, 01/11/2004, 14h00

Exercise 1: Let K be a field. Show, $I \trianglelefteq K[x]$ if and only if $\exists\, f \in K[x]$ such that $I = \langle f \rangle$.

Exercise 2: Let $0 \neq f = \sum_{|\alpha|=0}^{m} a_\alpha \underline{x}^\alpha \in R[x_1, \ldots, x_n]$ be a polynomial over the ring R. We call

$$\deg(f) := \max\{|\alpha| \mid a_\alpha \neq 0\}$$

the *degree* of f, and we set $\deg(0) = -\infty$. Show for $f, g \in R[x_1, \ldots, x_n]$:

a. $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$,

b. $\deg(f \cdot g) \leq \deg(f) + \deg(g)$,

c. $\deg(f \cdot g) = \deg(f) + \deg(g)$ if R contains no *zero-divisor* except 0.

Remark: $a \in R$ is called a *zero-divisor*, if there is a $0 \neq b \in R$ such that $a \cdot b = 0$.

Exercise 3: Let R be a ring and $I, J_1, \ldots, J_n \trianglelefteq R$. Show that

$$I : \left(\sum_{i=1}^{n} J_i \right) = \bigcap_{i=1}^{n} (I : J_i).$$

Exercise 4: Let R be a ring and $f = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$ a formal power series over R. Show:

a. f is a *unit* if and only if a_0 is a unit in R.

b. What are the units in $K[[x]]$ if K is a field?

c. If K is a field, then $0 \neq I \trianglelefteq K[[x]]$ if and only if there is some $n \geq 0$ such that $I = \langle x^n \rangle$.

Remark, $a \in R$ is called a *unit* if there exists an element $b \in R$ such that $a \cdot b = 1$.

Hint for a., consider first the case $a_0 = 1$ and recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.