Fachbereich Mathematik
Dr. Thomas Keilen

Winter Semester 2004/05, Set 2

Commutative Algebra

Due date: Monday, 15/11/2004, 14h00

Only exercises 1-3 should be handed in.

Exercise 5: Let R be a ring such that for every $r \in R$ there is an $n = n(r) > 1$ such that $r^n = r$.

a. Show that $\mathrm{Spec}(R) = \mathfrak{m} - \mathrm{Spec}(R)$.

b. Give an example of such a ring R which is not a field.

Exercise 6: Let $R \neq 0$ be a ring. Show that $\mathrm{Spec}(R)$ has a minimal element with respect to inclusion, i. e. $\exists\, P_0 \in \mathrm{Spec}(R) \;:\; \forall\, P \in \mathrm{Spec}(R)$ with $P \subseteq P_0$ we have $P = P_0$.

Exercise 7: Let R be a ring and $N(R)$ its nil-radical. Show the following are equivalent:

a. $R/N(R)$ is a field.

b. $|\mathrm{Spec}(R)| = 1$.

c. Every element of R is either unit or nilpotent.

Give an example for such a ring which is not a field.

Exercise 8: [Zariski Topology] – Will be discussed in the example class.

Let R be a ring. For a subset $A \subset R$ we define the *vanishing set* of A as

$$V(A) := \{P \in \mathrm{Spec}(R) \mid A \subseteq P\} \subseteq \mathrm{Spec}(R).$$

Show that the set $T := \{V(A) \mid A \subseteq R\}$ defines a topology on $\mathrm{Spec}(R)$ in the sense that T is the set of closed subsets of $\mathrm{Spec}(R)$.
To see this you should show the following:

a. $V(A) = V(\langle A \rangle) = V\big(\sqrt{\langle A \rangle}\big)$ for any $A \subseteq R$.
   In particular, $T = \{V(I) \mid I \trianglelefteq R\}$.

b. $V(0) = \mathrm{Spec}(R)$.

c. $V(R) = \emptyset$.

d. $V(I) \cup V(J) = V(I \cdot J) = V(I \cap J)$ for $I, J \trianglelefteq R$.

e. $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V\big(\sum_{\lambda \in \Lambda} I_\lambda\big)$ for $I_\lambda \trianglelefteq R$.