Fachbereich Mathematik — Winter Semester 2004/05, Set 3
Dr. Thomas Keilen

Commutative Algebra

Due date: Monday, 29/11/2004, 14h00

Only exercises 9, 10 and 11f. should be handed in.

Exercise 9: Let R be a ring, $\mathcal{M} = \{I \trianglelefteq R \mid I \text{ consists of zero-divisors}\}$. Show:

a. If $P \in \mathcal{M}$ is maximal w. r. t. inclusion, then P is prime.

b. If $I \in \mathcal{M}$, then there is a $P \in \mathcal{M}$ maximal such that $I \subset P$.

c. The set of zero-divisors in R is a union of prime ideals.

Exercise 10: Let R be an integral domain and $r \in R$.

a. If r is irreducible in R, then r is irreducible in $R[x]$.

b. If r is prime in R, then r is prime in $R[x]$.

Exercise 11: Let R be a ring and $X = \text{Spec}(R)$ endowed with the Zariski topology from Exercise 8. For $f \in R$ we define $X_f := X \setminus V(f) = \{P \in \text{Spec}(R) \mid f \notin P\}$.

a. $\{X_f \mid f \in R\}$ is a basis of the topology,
   i. e. if $U \subset X$ is open, then there are $f_\lambda \in R$, $\lambda \in \Lambda$, such that $U = \bigcup_{\lambda\in\Lambda} X_{f_\lambda}$.

b. If $f, g \in R$, then $X_f \cap X_g = X_{f\cdot g}$.

c. $X_f = \emptyset$ if and only if f is nilpotent.

d. $X_f = X$ if and only if f is a unit.

e. $X_f = X_g$ if and only if $\sqrt{\langle f\rangle} = \sqrt{\langle g\rangle}$.

f. X_f is quasi-compact, i. e. if $X_f = \bigcup_{\lambda\in\Lambda} U_\lambda$ with $U_\lambda \subset X$ open, then there are $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $X_f = \bigcup_{i=1}^n U_{\lambda_i}$.

Hint for part f.: Reduce to the case that $U_\lambda = X_{f_\lambda}$ and note that any linear combination of the f_λ only involves finitely many of them.

Exercise 12: Let $\varphi \in \text{Hom}(R, S)$ be a ring homomorphism, $X = \text{Spec}(R)$ and $Y = \text{Spec}(Y)$, both endowed with the Zariski topology.

a. If $Q \in \text{Spec}(S)$, then $Q^c = \varphi^{-1}(Q) \in \text{Spec}(R)$.
   In particular, $\varphi^* : Y \to X : Q \mapsto Q^c$ is defined.

b. If $I \trianglelefteq R$, then $(\varphi^*)^{-1}\big(V(I)\big) = V\big(I^e\big)$.
   In particular, φ^* is continuous.

c. If $f \in R$, then $(\varphi^*)^{-1}(X_f) = Y_{\varphi(f)}$.

d. If $J \trianglelefteq S$, then $\overline{\varphi^*\big(V(J)\big)} = V(J^c)$.

e. If φ is surjective, then φ^* is a homeomorphism onto its image $\text{Im}(\varphi^*) = V\big(\ker(\varphi)\big)$.

f. $\text{Im}(\varphi^*)$ is dense in X if and only if $\ker(\varphi) \subseteq \mathcal{N}(R)$.
   In particular, if φ is injective, then $\text{Im}(\varphi^*)$ is dense in X.