Fachbereich Mathematik
Dr. Thomas Keilen

Winter Semester 2004/05, Set 4

Commutative Algebra

Due date: Monday, 13/12/2004, 14h00

Exercise 13: Show that the following statements are equivalent:

a. R is a PID.

b. R is a UFD and for $a, b \in R$ with $1 \in \gcd(a, b)$ we have $1 \in \langle a, b\rangle$.

Hint, if R satisfies b. and $I \trianglelefteq R$ consider a maximal element $\langle a\rangle$ in the set $\mathcal{M} = \{\langle a\rangle \mid a \in I\}$ and show $I = \langle a\rangle$. Why does this maximal element exist?

Exercise 14: Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers and $M = R^3$. Consider the R-linear map $\varphi : M \to M : m \mapsto A \cdot m$ where

$$A = \begin{pmatrix} 1 + x^4 - x^7 + 3x^{100} & \cos(x) & 2 - \exp(x) \\ \sin(x) & \sum_{i=0}^{\infty}(-x)^i & \exp(\sin(x)) \\ x^4 - 5x^8 & \sum_{i=0}^{\infty}(5x + x^2)^i & 0 \end{pmatrix} \in \mathrm{Mat}(3 \times 3, R).$$

Is φ an isomorphism?

Exercise 15: Let R be a ring, M a finitely generated R-module and $\varphi \in \mathrm{Hom}_R(M, R^n)$ surjective. Show that $\ker(\varphi)$ is finitely generated as an R-module.

Hint, note that the short exact sequence $0 \to \ker(\varphi) \to M \to R^n \to 0$ is split exact.