**5.26 Corollary**

*Let $R$ be a Noetherian ring, $P \in \operatorname{Spec}(R)$ minimal over $a_1, \ldots, a_r \in R \setminus R^*$, then* $\operatorname{height}(P) \leq r$.

**Proof:** We do the proof by induction on $r$ where the case $r = 1$ follows from Krull's Principle Ideal Theorem 5.25.
Suppose therefore that there exists a chain of prime ideals

$$P_0 \subsetneqq P_1 \subsetneqq \ldots \subsetneqq P_{r+1} = P.$$

Let's consider first the case that $a_r \in P_1$. Then $P/\langle a_r \rangle$ is minimal over $\overline{a_1}, \ldots, \overline{a_{r-1}} \in R/\langle a_r \rangle$, and by induction hypothesis we have height $\big(P/\langle a_r \rangle \leq r - 1$. This, however, is in contradiction to the existence of the chain of prime ideals

$$P_1 \langle a_r \rangle \subsetneqq \ldots \subsetneqq P_{r+1} \langle a_r \rangle = P \langle a_r \rangle.$$

Let us now suppose the $a_r \notin P_1$. Choose $k \geq 2$ minimal such that $a_r \in P_k \setminus P_{k-1}$. Then

$$0 = P_{k-2}/P_{k-2} \subsetneqq P_{k-2} + \langle a_r \rangle / P_{k-2} \subseteq P_k / P_{k-2}.$$

Hence, $\operatorname{height}(P_k/P_{k-2}) \geq 2$, and by Krull's Principle Ideal Theorem 5.25 $P_k/P_{k-2}$ is not minimal over $\overline{a_r} \in R/P_{k-2}$. That is, there exists a $P'_{k-1} \in \operatorname{Spec}(R)$ such that $P_{k-2} \subsetneqq P'_{k-1} \subsetneqq P_k$ and $a_r \in P'_{k-1}$. This gives a chain of prime ideals

$$P_0 \subsetneqq P_1 \subsetneqq \ldots \subsetneqq P_{k-2} \subsetneqq P'_{k-1} \subsetneqq P_k \subsetneqq \ldots \subsetneqq P_{r+1} = P,$$

and $a_r \in P'_{k-1}$. We can of course repeat this process as long as $k-1 > 1$. Thus we end up with a chain of prime ideals

$$P_0 \subsetneqq P'_1 \subsetneqq \ldots \subsetneqq P'_{k-1} \subsetneqq \ldots \subsetneqq P_{r+1} = P$$

and $a_r \in P'_1$. Returning to the first case considered above this leads to a contradiction. $\square$