Fachbereich Mathematik
Dr. Thomas Markwig

Winter Semester 2005/06, Set 6

Commutative Algebra

Due date: Tuesday, 20/12/2005, 10h00

Exercise 21: Let $S \subseteq R$ be a multiplicatively closed subset, and consider the ring extension $i : R \to S^{-1}R : r \mapsto \frac{r}{1}$. Show that

$$\{P \in \text{Spec}(R) \mid S \cap P = \emptyset\} \longrightarrow \text{Spec}\left(S^{-1}R\right) : P \mapsto P^e = S^{-1}P$$

is bijective with inverse

$$\text{Spec}\left(S^{-1}R\right) \longrightarrow \{P \in \text{Spec}(R) \mid S \cap P = \emptyset\} : Q \mapsto Q^c = i^{-1}(Q).$$

In particular, for prime ideals $P \in \text{Spec}(R)$ we have $(P^e)^c = P$.

Exercise 22:

a. Let K be a field, $R = K[x, y, z]/\langle xz, yz\rangle$ and $P = \langle x, y, z-1\rangle \trianglelefteq R$. Show $R_P \cong K[z]_{\langle z-1\rangle}$.

b. Let R be a ring, $f \in R$ a non-zero-divisor. Show $R_f \cong R[x]/\langle fx - 1\rangle$.

Exercise 23: Let R be a ring and $\mathcal{N}(R)$ its nilradical. Show:

a. If $S \subseteq R$ multiplicatively closed, then $\mathcal{N}(S^{-1}R) = S^{-1}\mathcal{N}(R)$.

b. A ring is called *reduced* if it has no nilpotent elements except 0. Show that "being reduced" is a local property.

Exercise 24: Let $I := \langle 2, 1+\sqrt{-5}\rangle \lhd \mathbb{Z}\left[\sqrt{-5}\right]$. Show that I as an R-module is projective, but not free.

Hint, note that $2 \in I \cdot I$. Use this to show that $I \neq \langle x\rangle$ for any x, while for any prime P containing I we have I_P is generated by $1+\sqrt{-5}$. For the last statement use Nakayama's Lemma in a sensible way!

Exercise 25: **[Being flat is a local property.]**

Let R be a ring and M an R-module. Then the following are equivalent:

a. M is a flat R-module.

b. M_P is a flat R_P module for each $P \in \text{Spec}(R)$.

c. $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$ module for each $\mathfrak{m} \lhd \cdot R$.

Beweis: c.$\Longrightarrow$ a.: Let $N' \hookrightarrow N$ be R-linear and injective, then $N'_m \hookrightarrow N_m$ is R_m-linear and injective, since localisation is exact.. Then by Exercise 18,

$$\begin{array}{ccc}
(N' \otimes_R M)_m & \hookrightarrow & (N \otimes_R M)_m \\
\| & & \| \\
(N' \otimes_R M) \otimes_R R_m & & (N \otimes_R M) \otimes_R R_m \\
\| & & \| \\
(N' \otimes_R R_m) \otimes_{R_m} (M \otimes_R R_m) & & (N \otimes_R R_m) \otimes_{R_m} (M \otimes_R R_m) \\
\| & & \| \\
N'_m \otimes_{R_m} M_m & \hookrightarrow & N_m \otimes_{R_m} M_m
\end{array}$$

where the last row is injective by assumption, and thus so is the first row. Injectivity being a local property this implies that $N' \otimes_R M \hookrightarrow N \otimes_R M$ is injective.
a.$\Longrightarrow$ b.: Let $N' \hookrightarrow N$ be R_P-linear and injective, then $N' \hookrightarrow N$ is R-linear and injective. Note that if we consider N respectively N' as R-modules and localise w.r.t. P we get $N = N_P$ and $N' = N'_P$. Then by Exercise 18,

$$\begin{array}{ccc}
(N' \otimes_R M)_P & \hookrightarrow & (N \otimes_R M)_P \\
\| & & \| \\
(N' \otimes_R M) \otimes_R R_P & & (N \otimes_R M) \otimes_R R_P \\
\| & & \| \\
(N' \otimes_R R_P) \otimes_{R_P} (M \otimes_R R_P) & & (N \otimes_R R_P) \otimes_{R_P} (M \otimes_R R_P) \\
\| & & \| \\
N'_P \otimes_{R_P} M_P & & N_P \otimes_{R_P} M_P \\
\| & & \| \\
N' \otimes_{R_P} M_P & \hookrightarrow & N \otimes_{R_P} M_P
\end{array}$$

where the first row is injective by assumption and since localisation is exact, and thus so is the last row.
b.$\Longrightarrow$ c.: Clear. $\square$