Fachbereich Mathematik Dr. Thomas Markwig

Commutative Algebra

Due date: Tuesday, 20/12/2005, 10h00

Exercise 21: Let $S \subseteq R$ be a multiplicatively closed subset, and consider the ring extension $i: R \to S^{-1}R: r \mapsto \frac{r}{1}$. Show that

$$\left\{ \mathsf{P} \in \operatorname{Spec}(\mathsf{R}) \mid \mathsf{S} \cap \mathsf{P} = \emptyset \right\} \longrightarrow \operatorname{Spec}\left(\mathsf{S}^{-1}\mathsf{R}\right) : \mathsf{P} \mapsto \mathsf{P}^e = \mathsf{S}^{-1}\mathsf{P}$$

is bijective with inverse

$$\operatorname{Spec}(S^{-1}R) \longrightarrow \left\{ P \in \operatorname{Spec}(R) \mid S \cap P = \emptyset \right\} : Q \mapsto Q^{c} = \mathfrak{i}^{-1}(Q).$$

In particular, for prime ideals $P \in \operatorname{Spec}(R)$ we have $(P^e)^c = P$.

Exercise 22:

- a. Let K be a field, $R = K[x, y, z]/\langle xz, yz \rangle$ and $P = \langle x, y, z-1 \rangle \trianglelefteq R$. Show $R_P \cong K[z]_{\langle z-1 \rangle}$.
- b. Let R be a ring, $f \in R$ a non-zero-divisor. Show $R_f \cong R[x]/\langle fx 1 \rangle$.

Exercise 23: Let R be a ring and $\mathcal{N}(R)$ its nilradical. Show:

- a. If $S \subseteq R$ multiplicatively closed, then $\mathcal{N}(S^{-1}R) = S^{-1}\mathcal{N}(R)$.
- b. A ring is called *reduced* if it has no nilpotent elements except 0. Show that "being reduced" is a local property.

Exercise 24: Let $I := \langle 2, 1 + \sqrt{-5} \rangle \triangleleft \mathbb{Z}[\sqrt{-5}]$. Show that I as an R-module is projective, but not free.

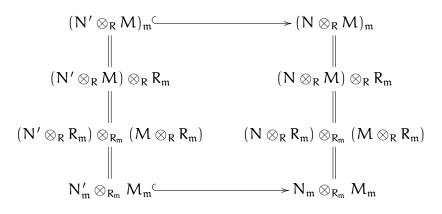
Hint, note that $2 \in I \cdot I$. Use this to show that $I \neq \langle x \rangle$ for any x, while for any prime P containing I we have I_P is generated by $1 + \sqrt{-5}$. For the last statement use Nakayama's Lemma in a sensible way!

Exercise 25: [Being flat is a local property.]

Let R be a ring and M an R-module. Then the following are equivalent:

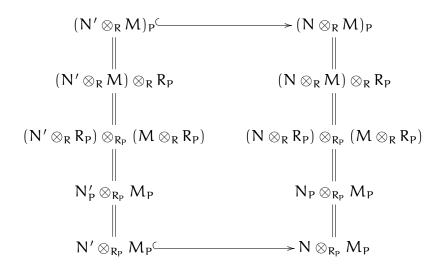
- a. M is a flat R-module.
- b. M_P is a flat R_P module for each $P \in \text{Spec}(R)$.
- c. $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ module for each $\mathfrak{m} \lhd \cdot R$.

Beweis: c. \Longrightarrow a.: Let $N' \hookrightarrow N$ be R-linear and injective, then $N'_{\mathfrak{m}} \hookrightarrow N_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$ -linear and injective, since localisation is exact. Then by Exercise 18,



where the last row is injective by assumption, and thus so is the first row. Injectivity being a local property this implies that $N' \otimes_R M \hookrightarrow N \otimes_R M$ is injective.

a. \Longrightarrow b.: Let N' \hookrightarrow N be R_P-linear and injective, then N' \hookrightarrow N is R-linear and injective. Note that if we consider N respectively N' as R-modules and localise w.r.t. P we get N = N_P and N' = N'_P. Then by Exercise 18,



where the first row is injective by assumption and since localisation is exact, and thus so is the last row.

 $b \Longrightarrow c$.: Clear.