Fachbereich Mathematik
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Winter Semester 2005/06, Set 12

Commutative Algebra

Due date: Tuesday, 14/02/2006, 10h00

Exercise 44: Prove the Algebraic HNS (Theorem 7.1) using Noether-Normalisation.

Exercise 45: Let R be a ring. Show that $\dim(R[x]) \geq \dim(R) + 1$.

Hint, consider ideals of the form $I[x] = \left\{ \sum_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in I \right\}$ for some ideal $I \trianglelefteq R$. – Note, if R is noetherian one can actually show equality, but that is much harder.

Exercise 46: Let R be an integral domain. Show:

a. R is a valuation ring if and only if for two ideals $I, J \trianglelefteq R$ we have $I \subseteq J$ or $J \subseteq I$.

b. If R is a valuation ring and $P \in \mathrm{Spec}(R)$, then R_P and R/P are valuation rings.

Exercise 47: [The field $K\{\{t\}\}$]

a. We call $A \subset \mathbb{R}$ *suitable* if A is infinite countable, bounded from below, and has no limit point, and we then set $\mathcal{A} := \{A \subset \mathbb{R} \mid A \text{ is suitable}\}$. Show that for $A, B \in \mathcal{A}$

$$A + B := A \cup B \in \mathcal{A} \quad \text{and} \quad A * B := \{a + b \mid a \in A, b \in B\} \in \mathcal{A}.$$

b. Let K be any field and consider the set

$$K\{\{t\}\} := \{f : \mathbb{R} \to K \mid \exists\, A \in \mathcal{A} \;:\; f(\alpha) = 0 \;\forall\, \alpha \notin A\}.$$

We define two binary operations on $K\{\{t\}\}$:

$$f + g : \mathbb{R} \to K : \alpha \mapsto f(\alpha) + g(\alpha)$$

and

$$f * g : \mathbb{R} \to K : \alpha \mapsto \sum_{\gamma \in \mathbb{R}} f(\alpha - \gamma) \cdot g(\gamma),$$

note for the latter that for a fixed α only finitely many summands are non-zero! Show that $(K\{\{t\}\}, +, *)$ is a field.

c. Show that $\mathrm{ord} : (K\{\{t\}\}^*, *) \to (\mathbb{R}, +) : f \mapsto \min\{\alpha \in \mathbb{R} \mid f(\alpha) \neq 0\}$ is a valuation.

d. R_{ord} is not noetherian, hence ord is not discrete, but $\dim(R_{\mathrm{ord}}) = 1$.

Hint for part b., show first that $(K\{\{t\}\}, +)$ is a subgroup of $(K^{\mathbb{R}}, +)$; the hard part is to show that every non-zero element of $K\{\{t\}\}$ has an inverse. For this consider first the case that $f(\alpha) = 0$ for $\alpha < 0$ and $f(0) = 1$, and use the geometric series. For part d., note that $\mathfrak{m}_{R_{\mathrm{ord}}} = \langle t^\alpha \mid \alpha > 0 \rangle$, where $t^\alpha : \mathbb{R} \to K$ satisfies $t^\alpha(\alpha) = 1$ and $t^\alpha(\beta) = 0$ for $\beta \neq \alpha$.

Remark 1

Let $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a sequence of real numbers. We define

$$\alpha_n \nearrow \infty \quad :\Longleftrightarrow \quad (\alpha_n)_{n\in\mathbb{N}} \text{ is strictly monotonously increasing and unbounded,}$$

and we set $\mathbb{A} := \{(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \alpha_n \nearrow \infty\}$. Obviously,

$$\Phi : \mathbb{A} \longrightarrow \mathcal{A} : (\alpha_n)_{n\in\mathbb{N}} \mapsto \{\alpha_n \mid n \in \mathbb{N}\}$$

is bijective.

For $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ we define

$$\sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n} : \mathbb{R} \longrightarrow K : \alpha \mapsto \begin{cases} a_n, & \text{if } \alpha = \alpha_n, \\ 0, & \text{else.} \end{cases}$$

That is, we use the "sequence" in order to store the values of a funciton in such a way, that the value at α_n is just the coefficient at t^{α_n}. Thus

$$\begin{aligned} K\{\{t\}\} &= \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \exists\, \alpha_n \nearrow \infty \;:\; f(\alpha) = 0 \;\forall\, \alpha \notin \{\alpha_n \mid n \in \mathbb{N}\} \right\} \\ &= \left\{ \sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n} \;\middle|\; \alpha_n \nearrow \infty, a_n \in K \right\}. \end{aligned}$$

Given $f = \sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n}, g = \sum_{n=0}^{\infty} b_n \cdot t^{\beta_n} \in K\{\{t\}\}$.

a. $f = g$ if and only if $a_n = b_m$ whenever $\alpha_n = \beta_m$ and if $a_i = b_j = 0$ if there is no matching.

b. $f*g = \sum_{n=0}^{\infty} \left(\sum_{\alpha_i+\beta_j=\gamma_n} a_i \cdot b_j \right) \cdot t^{\gamma_n}$, where $(\gamma_n)_{n\in\mathbb{N}} = \Phi^{-1}\big(\Phi((\alpha_n)_{n\in\mathbb{N}}) * \Phi((\beta_n)_{n\in\mathbb{N}})\big)$.

c. $f+g = \sum_{n=0}^{\infty} \big(f(\gamma_n) + g(\gamma_n)\big) \cdot t^{\gamma_n}$, where $(\gamma_n)_{n\in\mathbb{N}} = \Phi^{-1}\big(\Phi((\alpha_n)_{n\in\mathbb{N}}) + \Phi((\beta_n)_{n\in\mathbb{N}})\big)$.

d. If $\alpha_0 = 0$ and $a_0 = 1$, then $f^{-1} = \sum_{n=0}^{\infty} \left(-\sum_{k=1}^{\infty} a_k \cdot t^k \right)^n$.