Fachbereich Mathematik — Winter Semester 2007/08, Set 4

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Commutative Algebra

Due date: Friday, 16/11/2007, 14h00

Exercise 12: Let M be an R-module.

a. Prove that $\mu : M \to \mathrm{Hom}_R(R, M)$ with $\mu(m) : R \to M : r \mapsto r \cdot m$ is an isomorphism.

b. Give an example where $M \not\cong \mathrm{Hom}_R(M, R)$.

Exercise 13: Let R be an integral domain and $0 \neq I \trianglelefteq R$.
Show that I as R-module is free if and only if I is principal.

Exercise 14: Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers. Consider the R-linear map $\varphi : R^3 \to R^2 : m \mapsto A \cdot m$ where

$$A = \begin{pmatrix} 1 + x^4 - x^7 + 3x^{100} & \cos(x) & 2 - \exp(x) \\ x^4 - 5x^8 & \sum_{i=0}^{\infty}(5x + x^2)^i & 0 \end{pmatrix} \in \mathrm{Mat}(2 \times 3, R).$$

Is φ an epimorphism?

Exercise 15: Let $p \in \mathbb{Z}$ be a prime number. Consider the subring $R = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, p \not| b \right\} \leq \mathbb{Q}$ of the rational numbers, and consider $M = \mathbb{Q}$ as an R-module.

a. Show that R is local with maximal ideal $\mathfrak{m} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, p \not| b, p \mid a \right\}$.

b. $\mathfrak{m} \cdot M = M$, but $M \neq 0$.

c. Find a set of generators for M.