Fachbereich Mathematik
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Winter Semester 2007/08, Set 4

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# Commutative Algebra

Due date: Friday, 16/11/2007, 14h00

**Exercise 12:** Let $M$ be an $R$-module.

a. Prove that $\mu : M \to \mathrm{Hom}_R(R, M)$ with $\mu(m) : R \to M : r \mapsto r \cdot m$ is an isomorphism.

b. Give an example where $M \ncong \mathrm{Hom}_R(M, R)$.

**Exercise 13:** Let $R$ be an integral domain and $0 \neq I \trianglelefteq R$.
Show that $I$ as $R$-module is free if and only if $I$ is principal.

**Exercise 14:** Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers. Consider the $R$-linear map $\varphi : R^3 \to R^2 : m \mapsto A \cdot m$ where

$$A = \begin{pmatrix} 1 + x^4 - x^7 + 3x^{100} & \cos(x) & 2 - \exp(x) \\ x^4 - 5x^8 & \sum_{i=0}^{\infty} (5x + x^2)^i & 0 \end{pmatrix} \in \mathrm{Mat}(2 \times 3, R).$$

Is $\varphi$ an epimorphism?

**Exercise 15:** Let $p \in \mathbb{Z}$ be a prime number. Consider the subring $R = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, p \not| b \right\} \leq \mathbb{Q}$ of the rational numbers, and consider $M = \mathbb{Q}$ as an $R$-module.

a. Show that $R$ is local with maximal ideal $m = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, p \not| b, p \mid a \right\}$.

b. $m \cdot M = M$, but $M \neq 0$.

c. Find a set of generators for $M$.