Fachbereich Mathematik — Winter Semester 2007/08, Set 5
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Commutative Algebra

Due date: Friday, 23/11/2007, 14h00

Exercise 16: Let R be a ring, M a finitely generated R-module and $\varphi \in \mathrm{Hom}_R(M, R^n)$ surjective. Show that $\ker(\varphi)$ is finitely generated as an R-module.

Hint, note that the short exact sequence $0 \to \ker(\varphi) \to M \to R^n \to 0$ is split exact.

Exercise 17: Let R be a ring and P an R-module. Show that the following statements are equivalent:

a. If $\varphi \in \mathrm{Hom}_R(M, N)$ is surjective and $\psi \in \mathrm{Hom}_R(P, N)$, then there is a $\alpha \in \mathrm{Hom}_R(P, M)$ such that $\varphi \circ \alpha = \psi$, i.e.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\,\alpha"'] \arrow[d, "\psi"] \\ M \arrow[r, two heads, "\varphi"'] & N \end{tikzcd}$$

b. If $\varphi \in \mathrm{Hom}_R(M, N)$ is surjective, then $\varphi_* : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N) : \alpha \mapsto \varphi \circ \alpha$ is surjective.

c. If $0 \to M \to N \to P \to 0$ is exact, then it is split exact.

d. There is free module F and a submodule $M \leq F$ such that $P \oplus M \cong F$.

Exercise 18: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Show, if M' and M'' are finitely generated, then so is M.

Hint, you can do the proof using the Snake Lemma and the fact that a free module is projective. Alternatively you can simply write down a set of generators.

Exercise 19: Let R be a ring, M, M' and M'' R-modules, $\varphi \in \mathrm{Hom}_R(M', M)$ and $\psi \in \mathrm{Hom}_R(M, M'')$.
Show that

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

is exact if and only if for all R-modules P the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', P) \xrightarrow{\psi^*} \mathrm{Hom}_R(M, P) \xrightarrow{\varphi^*} \mathrm{Hom}_R(M', P)$$

is exact.