Fachbereich Mathematik
Thomas Markwig Winter Semester 2007/08, Set 6
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Commutative Algebra

Due date: Friday, 30/11/2007, 14h00

Exercise 20: Suppose that $(R, \mathfrak{m})$ is local ring and that $M \oplus R^m \cong R^n$ for some $n \geq m$. Show that then $M \cong R^{n-m}$.

Exercise 21: Let R' be an R-algebra and M and N be R-modules. Show that there is an isomorphism of R'-modules

$$\Phi : \big(M \otimes_R N\big) \otimes_R R' \longrightarrow \big(M \otimes_R R'\big) \otimes_{R'} \big(N \otimes_R R'\big) : m \otimes n \otimes r' \mapsto (m \otimes r') \otimes (n \otimes 1).$$

Recall that $M \otimes_R R'$ is an R'-module via $r' \cdot (m \otimes s') := m \otimes (r' \cdot s')$.

Exercise 22: Let $(R, \mathfrak{m})$ be a local ring, and M and N be finitely generated R-modules. Show that $M \otimes N = 0$ if and only if $M = 0$ or $N = 0$.

Hint, use Exercise 21 and Nakayama's Lemma.

Exercise 23: Let R be a ring, M and N be R-modules, and suppose $N = \langle n_\lambda \mid \lambda \in \Lambda \rangle$. Show:

a. $M \otimes_R N = \big\{ \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \;\big|\; m_\lambda \in M \text{ and only finitely many } m_\lambda \neq 0 \big\}$.

b. Let $x = \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \in M \otimes_R N$ with $m_\lambda \in M$ and only finitely many $m_\lambda \neq 0$.

   Then $x = 0$ if and only if there exist $m'_\theta \in M$ and $a_{\lambda,\theta} \in R$, $\theta \in \Theta$ some index set, such that
$$m_\lambda = \sum_{\theta \in \Theta} a_{\lambda,\theta} \cdot m'_\theta \quad \text{for all} \quad \lambda \in \Lambda$$
   and
$$\sum_{\lambda \in \Lambda} a_{\lambda,\theta} \cdot n_\lambda = 0 \quad \text{for all} \quad \theta \in \Theta.$$

Hint, for part b. consider first the case that N is free in the $(n_\lambda \mid \lambda \in \Lambda)$ and show that in that case actually all m_λ are zero. Then consider a free presentation $\bigoplus_{\theta \in \Theta} R \to \bigoplus_{\lambda \in \Lambda} R \to N \to 0$ of N and tensorize this with M.