Fachbereich Mathematik — Winter Semester 2007/08, Set 9
Thomas Markwig — Henning Meyer

Commutative Algebra

Due date: Friday, 21/12/2007, 14h00

Exercise 32:

a. Let $\varphi : R \to R'$ be a ring homomorphism and $Q \lhd R'$ a P-primary ideal. Show that $Q^c = \varphi^{-1}(Q)$ is $P^c = \varphi^{-1}(P)$-primary.

b. Let R be a ring, $P \in \mathrm{Spec}(R)$, and $n \geq 1$. Show that the *symbolic power* $P^{(n)} := \{a \in R \mid \exists\, s \in R \setminus P \, : \, s \cdot a \in P^n\}$ is a P-primary ideal.

Note, if $\iota : R \to R_P : a \mapsto \frac{a}{1}$, then $P^{(n)} = \left((P^n)^e\right)^c = \iota^{-1}\left(\langle P^n \rangle_{R_P}\right)$.

Exercise 33: Let R be a integral domain of dimension $\dim(R) = 1$, and let $0 \neq I \unlhd R$.

a. Show if $I = Q_1 \cap \ldots \cap Q_n$ is a minimal primary decomposition, then $I = Q_1 \cdots Q_n$.

b. If R is noetherian, then every non-zero ideal I is a finite product of primary ideals Q_i with $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$, and the factors are unique up to ordering.

Hint, Chinese Remainder Theorem.

Exercise 34: Find a minimal primary decomposition of $I = \langle 6 \rangle \lhd \mathbb{Z}\left[\sqrt{-5}\right]$.

Hint, consider the ideals $P = \langle 2, 1+\sqrt{-5} \rangle$, $Q = \langle 3, 1+\sqrt{-5} \rangle$, and $Q' = \langle 3, 1-\sqrt{-5} \rangle$.

Exercise 35: Let $R = K[x, y, z]$ for some field K.

a. Let $P = \langle x, y \rangle$ and $Q = \langle y, z \rangle$. Calculate a minimal primary decomposition of $I = P \cdot Q$. Which of the components are isolated, which are embedded?

b. Calculate a primary decomposition of $J = \langle xz - y^2, y - x^2 \rangle$.

Hint, in part b. consider $\varphi : R \to K[x]$ with $x \mapsto x, y \mapsto x^2, z \mapsto x^3$, $P = \ker(\varphi)$, and $Q = \langle x, y \rangle$. Show that $\ker(\varphi) = \langle y - x^2, z - x^3 \rangle$ using division with remainder.