Commutative Algebra

Due date: Friday, 31/01/2006, 14h00

Exercise 42 a., b. and c. need NOT be handed in for marking.

Exercise 40: Let $R \subset R'$ be an integral ring extension and let $\mathfrak{m}' \lhd \cdot R'$ be a maximal ideal such that $\mathfrak{m} = \mathfrak{m}' \cap R \lhd \cdot R$ is maximal as well. Is then $R'_{\mathfrak{m}'}$ integral over $R_{\mathfrak{m}}$?

Hint, consider $R = K[x^2 - 1]$, $R' = K[x]$, $\mathfrak{m}' = \langle x - 1 \rangle$, and $f = \frac{1}{1+x} \in R'_{\mathfrak{m}'}$.

Exercise 41: Let $R \subset R'$ be integral domains and $f, g \in R'[x]$ be monic polynomials. Show that if $f \cdot g \in \mathrm{Int}_{R'}(R)[x]$, then $f, g \in \mathrm{Int}_{R'}(R)[x]$.

Note, if we apply this to $R = \mathbb{Z}$ and $R' = \mathbb{Q}$, then we get for monic $f, g \in \mathbb{Q}[x]$ that $f \cdot g \in \mathbb{Z}[x]$ implies $f, g \in \mathbb{Z}[x]$.

Exercise 42: [Rings of Integers of Quadratic Number Fields]

Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree number (i.e. no square a^2 divides d), then $\mathbb{Q}\big[\sqrt{d}\big] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$ is a field extension of $\mathbb{Q}$ with $\dim_{\mathbb{Q}} \mathbb{Q}\big[\sqrt{d}\big] = 2$. Consider the *conjugation*

$$C : \mathbb{Q}\big[\sqrt{d}\big] \longrightarrow \mathbb{Q}\big[\sqrt{d}\big] : a + b\sqrt{d} \mapsto a - b\sqrt{d},$$

the *norm*

$$N : \mathbb{Q}\big[\sqrt{d}\big] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) \cdot C(a + b\sqrt{d}) = a^2 - b^2 d,$$

and the *trace*

$$T : \mathbb{Q}\big[\sqrt{d}\big] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) + C(a + b\sqrt{d}) = 2a.$$

Show:

a. $C(x \cdot y) = C(x) \cdot C(y)$ and $N(x \cdot y) = N(x) \cdot N(y)$ for $x, y \in \mathbb{Q}\big[\sqrt{d}\big]$.

b. C and T are $\mathbb{Q}$-linear.

c. If $x \in \mathbb{Q}\big[\sqrt{d}\big] \setminus \mathbb{Q}$, then $\mu_x = (t - x) \cdot \big(t - C(x)\big) = t^2 - T(x) \cdot t + N(x) \in \mathbb{Q}[t]$ is the minimal polynomial of x over $\mathbb{Q}$.

d. $x \in \mathbb{Q}\big[\sqrt{d}\big]$ is integral over $\mathbb{Z}$ if and only if $T(x)$ and $N(x)$ are integers.

e. $\mathrm{Int}_{\mathbb{Q}[\sqrt{d}]}(\mathbb{Z}) = \mathbb{Z}[\omega_d]$, where $\omega_d = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \mod 4 \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \mod 4. \end{cases}$