Fachbereich Mathematik — Winter Semester 2007/08, Set 12

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Commutative Algebra

Due date: Friday, 18/01/2007, 14h00

Exercise 43: Use Hilbert's Nullstellensatz and Exercise 38 to show that the dimension of the polynomial ring is $\dim(K[x_1, \ldots, x_n]) = n$.

Hint, for $\overline{K}[x_1, \ldots, x_n]$ reduce to the local case.

Exercise 44: Let $K \subseteq K'$ be a *field* extension, and let $T \subset K'$ (possibly infinite). T is called *algebraically independent* over K if every finite subset of T is algebraically independent over K. And an algebraically independent set T is a *transcendence basis* of K'/K if $T \cup \{t'\}$ is algebraically dependent for every $t' \in K' \setminus T$. Show:

a. An algebraically independent set T is a transcendence basis of K'/K if and only K' is integral over $K(T) = \left\{ \frac{f(t_1,\ldots,t_n)}{g(t_1,\ldots,t_n)} \mid f, g \in K[x_1, \ldots, x_n], t_1, \ldots, t_n \in T, n \geq 1 \right\}$.

b. If T and T' are transcendence bases of K'/K and $t \in T$, then there is a $t' \in T'$ such that $(T \setminus \{t\}) \cup \{t'\}$ is a transcendence basis of K'/K.

c. If T and T' are transcendence bases of K'/K, $|T| < \infty$, then $\operatorname{trdeg}_K(K') = |T| = |T'|$.

d. $\operatorname{trdeg}_K\big(K(x_1, \ldots, x_n)\big) = n$.

Hint for part b., if $T_0 = T \setminus \{t\}$, then consider the field extensions $K(T_0) \subset K'$, $K(T' \cup T_0) \subset K'$ and $K(T_0) \subset K(T' \cup T_0) = K(T_0)(T')$.

Which of these are integral (which is the same as algebraic)?

Exercise 45: Show that $\operatorname{trdeg}_K\big(K[x_1, \ldots, x_n]/\langle f \rangle\big) = n - 1$ for $f \in K[x_1, \ldots, x_n] \setminus K$.

Exercise 46: Let R be a finitely generated K-algebra which is an integral domain and let $K' = \operatorname{Quot}(R)$. Show that $\operatorname{trdeg}_K(R) = \operatorname{trdeg}_K(K')$.