Fachbereich Mathematik — Winter Semester 2009/10, Set 3
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Commutative Algebra

Due date: Friday, 13/11/2009, 14h00

Exercise 8: Let R be a ring such that for every $r \in R$ there is an $n = n(r) > 1$ such that $r^n = r$.

a. Show that $\operatorname{Spec}(R) = \mathfrak{m} - \operatorname{Spec}(R)$.

b. Give an example of such a ring R which is not a field.

Exercise 9: Let $R \neq 0$ be a ring. Show that $\operatorname{Spec}(R)$ has a minimal element with respect to inclusion, i. e. $\exists\, P_0 \in \operatorname{Spec}(R) \,:\, \forall\, P \in \operatorname{Spec}(R)$ with $P \subseteq P_0$ we have $P = P_0$.

Hint, use Zorn's Lemma with a suitable partial ordering on the set of all prime ideals.

Exercise 10: Let R be a ring and $N(R)$ its nil-radical. Show the following are equivalent:

a. $R/N(R)$ is a field.

b. $|\operatorname{Spec}(R)| = 1$.

c. Every element of R is either a unit or nilpotent.

Give an example for such a ring which is not a field.

Exercise 11: Let $d \in \mathbb{Z}$ be a squarefree, negative integer. Show that $\mathbb{Z}\left[\sqrt{d}\right]$ is a UFD if and only if $d \in \{-1, -2\}$.

Hint, show that 2 is not a prime, but if $d < -2$ it is irreducible. For the "non-primeness" note that either $2 \mid d$ or $2 \mid d-1$, and note that in $\mathbb{Q}\left[\sqrt{d}\right]$ every element is *uniquely* expressible as $a + b \cdot \sqrt{d}$ – why?

In-Class Exercise 7:

a. Find a prime factorisation of 11 in $\mathbb{Z}\left[\sqrt{-2}\right]$.
(Use known results for elementary number theory!)

b. $9 = 3 \cdot 3 = \left(1 + 2 \cdot \sqrt{-2}\right) \cdot \left(1 - 2 \cdot \sqrt{-2}\right)$.
How does this fit with the result from Exercise 11?

In-Class Exercise 8: Which of the following ideals I in $\mathbb{Z}[x]$ is a maximal ideal?

a. $I = \langle 5, 11x^3 + x - 1 \rangle$.

b. $I = \langle 4, x^2 + x + 1, x^2 + x - 1 \rangle$.

How many elements does the corresponding field $\mathbb{Z}[x]/I$ have?

In-Class Exercise 9: Let K be any field. Show that $x^2 - y^3 \in K[x, y]$ is irreducible.