

## Commutative Algebra

Due date: Friday, 21/10/11, 14h00

**Exercise 1:** Let  $0 \neq f = \sum_{|\alpha|=0}^m a_\alpha x^\alpha \in R[x_1, \dots, x_n]$  be a polynomial over the ring  $R$ . Recall:

$$\deg(f) := \max\{|\alpha| \mid a_\alpha \neq 0\}$$

is the *degree* of  $f$ , and we set  $\deg(0) = -\infty$ . Show for  $f, g \in R[x_1, \dots, x_n]$ :

- $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$ ,
- $\deg(f \cdot g) \leq \deg(f) + \deg(g)$ ,
- $\deg(f \cdot g) = \deg(f) + \deg(g)$ , if  $R$  is an integral domain.

Note,  $R$  is an integral domain if  $r \cdot r' = 0$  for  $r, r' \in R$  implies that  $r = 0$  or  $r' = 0$ .

**Exercise 2:** Let  $K$  be a ring,  $d \in \mathbb{N}$ , and

$$K[x_1, \dots, x_n]_d = \left\{ \sum_{|\alpha|=\alpha_1+\dots+\alpha_n=d} a_\alpha \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid a_\alpha \in K \right\}.$$

We call the elements of  $K[x_1, \dots, x_n]_d$  homogeneous of degree  $d$ .

- Show that every polynomial  $0 \neq f \in K[x_1, \dots, x_n]$  of degree  $d$  admits a unique decomposition  $f = f_0 + \dots + f_d$  with  $f_i \in K[x_1, \dots, x_n]_i$ . We call the  $f_i$  the *homogeneous summands* of  $f$ .
- An ideal  $I \trianglelefteq K[x_1, \dots, x_n]$  is called *homogeneous*, if  $f \in I$  implies that the homogeneous summands of  $f$  belong to  $I$ .

Show that  $I$  is homogeneous if and only if  $I$  is generated by homogeneous elements.

**Exercise 3:** [The field  $K\{\{t\}\}$ ]

- We call  $A \subset \mathbb{R}$  *suitable* if  $A$  is infinite countable, bounded from below, and has no limit point, and we then set  $\mathcal{A} := \{A \subset \mathbb{R} \mid A \text{ is suitable}\}$ . Show that for  $A, B \in \mathcal{A}$

$$A + B := A \cup B \in \mathcal{A} \quad \text{and} \quad A * B := \{a + b \mid a \in A, b \in B\} \in \mathcal{A}.$$

b. Let  $K$  be any field and consider the set

$$K\{\{t\}\} := \{f : \mathbb{R} \rightarrow K \mid \exists A \in \mathcal{A} : f(\alpha) = 0 \forall \alpha \notin A\}.$$

We define two binary operations on  $K\{\{t\}\}$ :

$$f + g : \mathbb{R} \rightarrow K : \alpha \mapsto f(\alpha) + g(\alpha)$$

and

$$f * g : \mathbb{R} \rightarrow K : \alpha \mapsto \sum_{\gamma \in \mathbb{R}} f(\alpha - \gamma) \cdot g(\gamma),$$

note for the latter that for a fixed  $\alpha$  only finitely many summands are non-zero!

Show that  $(K\{\{t\}\}, +, *)$  is a field.

Hint for part b., show first that  $(K\{\{t\}\}, +)$  is a subgroup of  $(K^{\mathbb{R}}, +)$ . The hard part is to show that every non-zero element of  $K\{\{t\}\}$  has an inverse. For this consider first the case that  $f(\alpha) = 0$  for  $\alpha < 0$  and  $f(0) = 1$ , and use the geometric series.

### Remark 1

Let  $(\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$  be a sequence of real numbers. We define

$$\alpha_n \nearrow \infty \iff (\alpha_n)_{n \in \mathbb{N}} \text{ is strictly monotonously increasing and unbounded,}$$

and we set  $\Lambda := \{(\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \alpha_n \nearrow \infty\}$ . Obviously,

$$\Phi : \Lambda \longrightarrow \mathcal{A} : (\alpha_n)_{n \in \mathbb{N}} \mapsto \{\alpha_n \mid n \in \mathbb{N}\}$$

is bijective.

For  $(\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$  and  $(a_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$  we define

$$\sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n} : \mathbb{R} \longrightarrow K : \alpha \mapsto \begin{cases} a_n, & \text{if } \alpha = \alpha_n, \\ 0, & \text{else.} \end{cases}$$

That is, we use the “sequence” in order to store the values of a function in such a way, that the value at  $\alpha_n$  is just the coefficient at  $t^{\alpha_n}$ . Thus

$$\begin{aligned} K\{\{t\}\} &= \{f : \mathbb{R} \rightarrow K \mid \exists \alpha_n \nearrow \infty : f(\alpha) = 0 \forall \alpha \notin \{\alpha_n \mid n \in \mathbb{N}\}\} \\ &= \left\{ \sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n} \mid \alpha_n \nearrow \infty, a_n \in K \right\}. \end{aligned}$$

Given  $f = \sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n}$ ,  $g = \sum_{n=0}^{\infty} b_n \cdot t^{\beta_n} \in K\{\{t\}\}$ .

- $f = g$  if and only if  $a_n = b_m$  whenever  $\alpha_n = \beta_m$  and if  $a_i = b_j = 0$  if there is no matching.
- $f * g = \sum_{n=0}^{\infty} \left( \sum_{\alpha_i + \beta_j = \gamma_n} a_i \cdot b_j \right) \cdot t^{\gamma_n}$ , where  $(\gamma_n)_{n \in \mathbb{N}} = \Phi^{-1}(\Phi((\alpha_n)_{n \in \mathbb{N}}) * \Phi((\beta_n)_{n \in \mathbb{N}}))$ .
- $f + g = \sum_{n=0}^{\infty} (f(\gamma_n) + g(\gamma_n)) \cdot t^{\gamma_n}$ , where  $(\gamma_n)_{n \in \mathbb{N}} = \Phi^{-1}(\Phi((\alpha_n)_{n \in \mathbb{N}}) + \Phi((\beta_n)_{n \in \mathbb{N}}))$ .
- If  $\alpha_0 = 0$  and  $a_0 = 1$ , then  $f^{-1} = \sum_{n=0}^{\infty} \left( - \sum_{k=1}^{\infty} a_k \cdot t^k \right)^n$ .