Commutative Algebra

Due date: Monday, 7/11/2011, 10h00

Exercise 8: Let R be a ring such that for every $r \in R$ there is an $n = n(r) > 1$ such that $r^n = r$.

a. Show that $\mathrm{Spec}(R) = \mathfrak{m} - \mathrm{Spec}(R)$.

b. Give an example of such a ring R which is not a field.

Exercise 9: Let $R \neq 0$ be a ring. Show that $\mathrm{Spec}(R)$ has a minimal element with respect to inclusion, i. e. $\exists\, P_0 \in \mathrm{Spec}(R)\ :\ \forall\, P \in \mathrm{Spec}(R)$ with $P \subseteq P_0$ we have $P = P_0$.

Hint, use Zorn's Lemma with a suitable partial ordering on the set of all prime ideals.

Exercise 10: Let R be a ring and $N(R)$ its nil-radical. Show the following are equivalent:

a. $R/N(R)$ is a field.

b. $|\mathrm{Spec}(R)| = 1$.

c. Every element of R is either a unit or nilpotent.

Give an example for such a ring which is not a field.

Exercise 11: Let M be an R-module.

a. Prove that $\mu : M \to \mathrm{Hom}_R(R, M)$ with $\mu(m) : R \to M : r \mapsto r\cdot m$ is an isomorphism.

b. Give an example where $M \ncong \mathrm{Hom}_R(M, R)$.

Exercise 12: Let R be an integral domain and $0 \neq I \trianglelefteq R$.
Show that I as R-module is free if and only if I is principal.

In-Class Exercise 7: Which of the following ideals I in $\mathbb{Z}[x]$ is a maximal ideal?

a. $I = \langle 5, 11x^3 + x - 1\rangle$.

b. $I = \langle 4, x^2 + x + 1, x^2 + x - 1\rangle$.

How many elements does the corresponding field $\mathbb{Z}[x]/I$ have?

In-Class Exercise 8: Let K be any field. Show that $x^2 - y^3 \in K[x, y]$ is irreducible.