Fachbereich Mathematik — Winter Semester 2011/12, Set 5
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Commutative Algebra

Due date: Monday, 21/11/2011, 10h00

Exercise 17: Suppose that $(R,\mathfrak{m})$ is local ring and that $M\oplus R^m\cong R^n$ for some $n\geq m$. Show that then $M\cong R^{n-m}$.

Exercise 18: Let R' be an R-algebra and M and N be R-modules. Show that there is an isomorphism of R'-modules

$$\Phi:\big(M\otimes_R N\big)\otimes_R R'\longrightarrow\big(M\otimes_R R'\big)\otimes_{R'}\big(N\otimes_R R'\big):m\otimes n\otimes r'\mapsto(m\otimes r')\otimes(n\otimes 1).$$

Recall that $M\otimes_R R'$ is an R'-module via $r'\cdot(m\otimes s'):=m\otimes(r'\cdot s')$.

Exercise 19: Let $(R,\mathfrak{m})$ be a local ring, and M and N be finitely generated R-modules. Show that $M\otimes N=0$ if and only if $M=0$ or $N=0$.

Hint, use Exercise 18 and Nakayama's Lemma.

Exercise 20: Let R be a ring, M and N be R-modules, and suppose $N=\langle n_\lambda\mid\lambda\in\Lambda\rangle$. Show:

a. $M\otimes_R N=\big\{\sum_{\lambda\in\Lambda}m_\lambda\otimes n_\lambda\;\big|\;m_\lambda\in M\text{ and only finitely many }m_\lambda\neq 0\big\}$.

b. Let $x=\sum_{\lambda\in\Lambda}m_\lambda\otimes n_\lambda\in M\otimes_R N$ with $m_\lambda\in M$ and only finitely many $m_\lambda\neq 0$.

   Then $x=0$ if and only if there exist $m'_\theta\in M$ and $a_{\lambda,\theta}\in R$, $\theta\in\Theta$ some index set, such that

$$m_\lambda=\sum_{\theta\in\Theta}a_{\lambda,\theta}\cdot m'_\theta\quad\text{for all}\quad\lambda\in\Lambda$$

   and

$$\sum_{\lambda\in\Lambda}a_{\lambda,\theta}\cdot n_\lambda=0\quad\text{for all}\quad\theta\in\Theta.$$

Hint, for part b. consider first the case that N is free in the $(n_\lambda\mid\lambda\in\Lambda)$ and show that in that case actually all m_λ are zero. Then consider a free presentation $\bigoplus_{\theta\in\Theta}R\to\bigoplus_{\lambda\in\Lambda}R\to N\to 0$ of N and tensorize this with M.

In-Class Exercise 11:

a. Consider the $\mathbb{Z}$-modules $M=\mathbb{Z}/2\mathbb{Z}$ and $N=\mathbb{Z}/4\mathbb{Z}$. How many elements does $M\otimes_{\mathbb{Z}}N$ have? Is it isomorphic to a $\mathbb{Z}$-module that you know?

b. Consider the $\mathbb{Z}$-module $M=\mathbb{Z}^3\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/5\mathbb{Z}$ and the $\mathbb{Q}$-vector space $M\otimes_{\mathbb{Z}}\mathbb{Q}$. What is its dimension?

In-Class Exercise 12: Let K be a field. Is the K-vector space $K[x]\otimes_K K[y]$ isomorphic to a K-vector space that you know very well? Can you define a multiplication on the tensor product, such that it becomes a K-algebra that you know?