Fachbereich Mathematik — Winter Semester 2011/12, Set 09
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Commutative Algebra

Due date: Monday, 19/12/2011, 10h00

Exercise 33: Let R be a noetherian integral domain, which is not a field, such that each ideal is a finite product of prime ideals.
Show that R is a PID if and only if R is a UFD and $\dim(R) = 1$.

Note, a noetherian ID of dimension one where each ideal is a product of primes is called a Dedekind domain and has by Exercise 30 "unique prime factorisation" for ideals!

Exercise 34: Give an example of a ring R with two "maximal" chains of prime ideals of different length.

Exercise 35: Let K be a field and $\overline{K}$ its algebraic closure and let $f \in K[x_1, \ldots, x_n]$.

a. Show that $\overline{K}[x_1, \ldots, x_n]$ is integral over $K[x_1, \ldots, x_n]$.

b. Show that $f \cdot \overline{K}[x_1, \ldots, x_n] \cap K[x_1, \ldots, x_n] = f \cdot K[x_1, \ldots, x_n]$.

c. Show that $\overline{K}[x_1, \ldots, x_n]/\langle f\rangle$ is integral over $K[x_1, \ldots, x_n]/\langle f\rangle$.

Hint, in part b. one may use the monomial ordering from Exercise 36 c.

Exercise 36: [Rings of Invariants]

Let G be a *finite* group and $R = K\big[\underline{x}\big]/I$ a finitely generated K-algebra, $G \to \mathrm{Aut}_{K-alg}(R)$ a group homomorphism (we say that G *acts* on R via K-algebra automorphisms), and write $g \cdot f := \alpha(g)(f)$ for $g \in G$ and $f \in R$. Moreover, consider $R^G = \{f \in R \mid g \cdot f = f\ \forall\ g \in G\}$, the *ring of invariants of* G *in* R.

a. Show that R is integral over R^G.

b. Show that R^G is a finitely generated K-algebra, hence noetherian.

c. Let $\mathrm{Mon}(\underline{x}) = \{0\} \cup \{\underline{x}^\alpha \mid \alpha \in \mathbb{N}^n\}$, $\mathrm{Mon}(f) = \{\underline{x}^\alpha \mid a_\alpha \neq 0\}$ for $0 \neq f = \sum_\alpha a_\alpha \underline{x}^\alpha \in K[\underline{x}]$ and $\mathrm{Mon}(0) = \{0\}$. We define a *well-ordering* on $\mathrm{Mon}(\underline{x})$ by $\underline{x}^\alpha > 0$ for all α and

$$\underline{x}^\alpha > \underline{x}^\beta \iff \begin{array}{l} \deg(\underline{x}^\alpha) > \deg(\underline{x}^\beta) \quad \text{or} \\ (\deg(\underline{x}^\alpha) = \deg(\underline{x}^\beta) \quad \text{and} \quad \exists\ i\ : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i), \end{array}$$

and we call $\mathrm{lm}(f) = \max\big(\mathrm{Mon}(f)\big)$ the *leading monomial of* f.
Show, $(\underline{x}^\alpha > \underline{x}^\beta \implies \underline{x}^\alpha \cdot \underline{x}^\gamma > \underline{x}^\beta \cdot \underline{x}^\gamma)$, and thus $\mathrm{lm}(f \cdot g) = \mathrm{lm}(f) \cdot \mathrm{lm}(g)$.

d. Consider the group homomorphism

$$\mathrm{Sym}(n) \longrightarrow \mathrm{Aut}_{K-alg}\big(K[x_1, \ldots, x_n]\big) : \sigma \mapsto \big(f \mapsto f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\big),$$

and the polynomial $(X + x_1) \cdots (X + x_n) = X^n + s_1 X^{n-1} + \ldots + s_n \in K[x_1, \ldots, x_n][X]$.
Show, $K[x_1, \ldots, x_n]^{\mathrm{Sym}(n)} = K[s_1, \ldots, s_n]$.

Hint, use Exercise 28 to solve part b., for part d. show first that $\underline{x}^\alpha = \mathrm{lm}(f)$ for $f \in K[x_1, \ldots, x_n]^{\mathrm{Sym}(n)}$ implies $\alpha_1 \geq \ldots \geq \alpha_n$, and deduce that there is a $g \in K[s_1, \ldots, s_n]$ such that $\mathrm{lm}(f) = \mathrm{lm}(g)$. Use this to do induction on $\mathrm{lm}(f)$ in order to show that actually $f \in K[s_1, \ldots, s_n]$. Note that $s_i = \sum_{1 \leq j_1 < \ldots < j_i \leq n} x_{j_1} \cdots x_{j_i}$, so what is $\mathrm{lm}(s_i)$?

In-Class Exercise 20: Let $R = K[x, y, z]_{\langle x,y,z\rangle}$, $I = \langle x^2 - y^2, xz - y\rangle$, $J = \langle x^2 - y^2, xz - yz\rangle$.
Compute $\dim(R/I)$ and $\dim(R/J)$.