

## Commutative Algebra

Due date: Monday, 18/01/2012, 10h00

### Exercise 40:

- Show that every maximal ideal in  $\mathbb{Z}[x]$  can be generated by a prime number  $p \in \mathbb{Z}$  and a polynomial  $f \in \mathbb{Z}[x]$  such that its residue class in  $\mathbb{Z}_p[x]$  is irreducible.
- Show that  $\dim(\mathbb{Z}[x]) = 2$ .

Hint, show in a. first that a maximal ideal cannot be principal, then that it contains a prime number and finally that modulo that prime number it will be generated by a single polynomial. Recall that  $\mathbb{Z}[x]/\langle p \rangle \cong \mathbb{Z}_p[x]$  is a PID and that in  $\mathbb{Q}[x]$  the Bézout identity holds.

**Exercise 41:** Let  $K \subseteq K'$  be a field extension, and let  $T \subset K'$  (possibly infinite).  $T$  is called *algebraically independent* over  $K$  if every finite subset of  $T$  is algebraically independent over  $K$ . And an algebraically independent set  $T$  is a *transcendence basis* of  $K'/K$  if  $T \cup \{t'\}$  is algebraically dependent for every  $t' \in K' \setminus T$ . Show:

- An algebraically independent set  $T$  is a transcendence basis of  $K'/K$  if and only if  $K'$  is integral over  $K(T)$ .

$$K(T) = \left\{ \frac{f(t_1, \dots, t_n)}{g(t_1, \dots, t_n)} \mid f, g \in K[x_1, \dots, x_n], g \neq 0, t_1, \dots, t_n \in T, n \geq 1 \right\}.$$

- If  $T$  and  $T'$  are transcendence bases of  $K'/K$  and  $t \in T$ , then there is a  $t' \in T'$  such that  $(T \setminus \{t\}) \cup \{t'\}$  is a transcendence basis of  $K'/K$ .
- If  $T$  and  $T'$  are transcendence bases of  $K'/K$ ,  $|T| < \infty$ , then  $\text{trdeg}_K(K') = |T| = |T'|$ .
- $\text{trdeg}_K(K(x_1, \dots, x_n)) = n$ .

Hint for part b., if  $T_0 = T \setminus \{t\}$ , then consider the field extensions  $K(T_0) \subset K'$ ,  $K(T' \cup T_0) \subset K'$  and  $K(T_0) \subset K(T' \cup T_0) = K(T_0)(T')$ .

Which of these are integral (which is the same as algebraic)?

**Exercise 42:** Find a Noether normalisation of  $R = K[x, y]/\langle x^3 - y^2 \rangle$  and compute the normalisation of  $R$ .

**Exercise 43:** Let  $R$  be a finitely generated  $K$ -algebra which is an integral domain and let  $K' = \text{Quot}(R)$ . Show that:

- If  $\beta_1, \dots, \beta_d \in R$  are algebraically independent over  $K$  and  $R$  is algebraic over  $K[\beta_1, \dots, \beta_d]$ , then  $\text{Quot}(R)$  is algebraic over  $K(\beta_1, \dots, \beta_d)$ .
- $\text{trdeg}_K(R) = \text{trdeg}_K(K')$ .

**In-Class Exercise 22:** Find all maximal ideals in  $\mathbb{C}[x, y]/\langle x^3 - x^2, x^2y - 2x^2 \rangle$ .

**In-Class Exercise 23:** Compute  $\text{Quot}(R)$  and  $\dim(\text{Quot}(R))$  for  $R = K[x, y]/\langle x^2, xy \rangle$ .