

Commutative Algebra

Due date: Monday, 25/01/2012, 10h00

Exercise 44: Prove the Algebraic HNS (Theorem 7.1) using Noether-Normalisation.

Exercise 45: Let R be a ring. Show that $\dim(R[x]) \geq \dim(R) + 1$.

Hint, consider ideals of the form $I[x] = \{ \sum_{i=0}^n a_i x^i \mid n \geq 0, a_i \in I \}$ for some ideal $I \subseteq R$. – Note, if R is noetherian one can actually show equality, but that is much harder.

Exercise 46: Let R be an integral domain. Show:

- R is a valuation ring if and only if for two ideals $I, J \subseteq R$ we have $I \subseteq J$ or $J \subseteq I$.
- If R is a valuation ring and $P \in \text{Spec}(R)$, then R_P and R/P are valuation rings.

Exercise 47: [A valuation on the field $K\{\{t\}\}$]

Let $K\{\{t\}\}$ be the field from Exercise 3.

- Show that $\text{ord} : (K\{\{t\}\}^*, *) \rightarrow (\mathbb{R}, +) : f \mapsto \min\{\alpha \in \mathbb{R} \mid f(\alpha) \neq 0\}$ is a valuation.
- R_{ord} is not noetherian, hence ord is not discrete, but $\dim(R_{\text{ord}}) = 1$.
- If $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ are algebraically independent over \mathbb{Q} , then $(t^{\alpha_1}, \dots, t^{\alpha_n})$ are algebraically independent over K . In particular, $\text{trdeg}_K(K\{\{t\}\}) = \infty$.

Hint for part b., note that $m_{R_{\text{ord}}} = \langle t^\alpha \mid \alpha > 0 \rangle$, where $t^\alpha : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $t^\alpha(\alpha) = 1$ and $t^\alpha(\beta) = 0$ for $\beta \neq \alpha$.

In-Class Exercise 24: Compute the dimension of $K[x, y, z]_{\langle x^2 - yz \rangle}$.

In-Class Exercise 25: Is the ring $\mathbb{C}[x, y] / \langle x^2 - y^2 - y^3 \rangle$ a Dedekind domain?

In-Class Exercise 26: Is the ring $\mathbb{C}[x, y] / \langle x - y^2 - y^3 \rangle$ a Dedekind domain?