

## Commutative Algebra

Due date: Monday, 10/11/2014, 10h00

The in-class exercises need not be handed in for marking. They should be discussed in class. No rigorous proofs are expected for these.

**Exercise 4:** Let  $R$  be a ring. Obviously  $R \hookrightarrow R[x_1, \dots, x_n] : a \mapsto a$  is a ring homomorphism and thus makes  $R[x_1, \dots, x_n]$  an  $R$ -algebra.

- Show that  $R[x_1, \dots, x_n]$  satisfies the following universal property: if  $(R', \varphi)$  is any  $R$ -algebra and  $a_1, \dots, a_n \in R'$  are given, then there is a unique  $R$ -algebra homomorphism  $\alpha : R[x_1, \dots, x_n] \rightarrow R'$  such that  $\alpha(x_i) = a_i$  for all  $i = 1, \dots, n$ .
- Let  $I \trianglelefteq R[x_1, \dots, x_n]$  and  $J \trianglelefteq R[y_1, \dots, y_m]$ . Show that the following are equivalent:
  - $\varphi : R[x_1, \dots, x_n]/I \rightarrow R[y_1, \dots, y_m]/J$  is an  $R$ -algebra homomorphism
  - There are  $f_1, \dots, f_n \in R[y_1, \dots, y_m]$  such that  $g(f_1, \dots, f_n) \in J$  for all  $g \in I$  and  $\varphi(\overline{g}) = \overline{g(f_1, \dots, f_n)}$  for all  $\overline{g} \in R[x_1, \dots, x_n]/I$ .
  - There is an  $R$ -algebra homomorphism  $\psi : R[x_1, \dots, x_n] \rightarrow R[y_1, \dots, y_m]$  such that  $\psi(I) \subseteq J$  and  $\varphi(\overline{g}) = \overline{\psi(g)}$ .

Note, a. means: we may uniquely define an  $R$ -algebra homomorphism on  $R[x_1, \dots, x_n]$  by just specifying the images of the  $x_i$ !

**Exercise 5:** Let  $R$  be a ring and  $I, J_1, \dots, J_n \trianglelefteq R$ . Show that:

- $I : (\sum_{i=1}^n J_i) = \bigcap_{i=1}^n (I : J_i)$ .
- $(\bigcap_{i=1}^n J_i) : I = \bigcap_{i=1}^n (J_i : I)$ .
- $\sqrt{J_1 \cap \dots \cap J_n} = \sqrt{J_1} \cap \dots \cap \sqrt{J_n}$ .
- $\sqrt{J_1 + \dots + J_n} \supseteq \sqrt{J_1} + \dots + \sqrt{J_n}$ .

**Exercise 6:** Let  $R$  be a ring and  $f = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$  a formal power series over  $R$ . Show:

- $f$  is a *unit* if and only if  $a_0$  is a unit in  $R$ .
- What are the units in  $K[[x]]$  if  $K$  is a field?
- $x$  is not a zero-divisor in  $R[[x]]$ .
- If  $f$  is nilpotent, then  $a_n$  is nilpotent for all  $n$ . Is the converse true?

Hint for a., consider first the case  $a_0 = 1$  and recall that  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ .

**In-Class Exercise 4:** Consider the ring extension

$$\iota: \mathbb{Z} \longrightarrow \mathbb{Z}_7 = \left\{ \frac{z}{7^n} \mid n \geq 0, z \in \mathbb{Z} \right\} : z \mapsto z$$

and the ideals  $I = \langle 84 \rangle \triangleleft \mathbb{Z}$  and  $J = \langle 15 \rangle \triangleleft \mathbb{Z}_7$ . Give generators  $I^e$ ,  $I^{ec}$ ,  $J^c$ , and  $J^{ce}$ .

**In-Class Exercise 5:** Does the following equality of ideals hold in the polynomial ring  $\mathbb{C}[x, y]$ :

$$\langle x^3 - x^2, x^2y - x^2, xy - y, y^2 - y \rangle = \langle x^2, y \rangle \cap \langle x - 1, y - 1 \rangle.$$

**In-Class Exercise 6:** What are the prime ideals in  $\mathbb{C}[x, y]$  containing the ideal  $I = \langle x^2y - x^2 \rangle$ .