Fachbereich Mathematik
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Winter Semester 2014/15, Set 3

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# Commutative Algebra

Due date: Monday, 17/11/2014, 10h00

**Exercise 8:** Let $R$ be a ring such that for every $r \in R$ there is an $n = n(r) > 1$ such that $r^n = r$.

a. Show that $\text{Spec}(R) = \mathfrak{m} - \text{Spec}(R)$.

b. Give an example of such a ring $R$ which is not a field.

**Exercise 9:** Let $R \neq 0$ be a ring. Show that $\text{Spec}(R)$ has a minimal element with respect to inclusion, i. e. $\exists\, P_0 \in \text{Spec}(R) \,:\, \forall\, P \in \text{Spec}(R)$ with $P \subseteq P_0$ we have $P = P_0$.

Hint, use Zorn's Lemma with a suitable partial ordering on the set of all prime ideals.

**Exercise 10:** Let $R$ be a ring and $N(R)$ its nil-radical. Show the following are equivalent:

a. $R/N(R)$ is a field.

b. $|\text{Spec}(R)| = 1$.

c. Every element of $R$ is either a unit or nilpotent.

Give an example for such a ring which is not a field.

**Exercise 11:** Let $M$ be an $R$-module.

a. Prove that $\mu : M \to \text{Hom}_R(R, M)$ with $\mu(m) : R \to M : r \mapsto r \cdot m$ is an isomorphism.

b. Give an example where $M \not\cong \text{Hom}_R(M, R)$.

**Exercise 12:** Let $R$ be an integral domain and $0 \neq I \trianglelefteq R$.
Show that $I$ as $R$-module is free if and only if $I$ is principal.

**In-Class Exercise 7:** Which of the following ideals $I$ in $\mathbb{Z}[x]$ is a maximal ideal?

a. $I = \langle 5, 11x^3 + x - 1 \rangle$.

b. $I = \langle 4, x^2 + x + 1, x^2 + x - 1 \rangle$.

How many elements does the corresponding field $\mathbb{Z}[x]/I$ have?

**In-Class Exercise 8:** Let $K$ be any field. Show that $x^2 - y^3 \in K[x, y]$ is irreducible.