Fachbereich Mathematik — Winter Semester 2014/15, Set 4

Thomas Markwig — Cornelia Rottner

Commutative Algebra

Due date: Monday, 24/11/2014, 10h00

Exercise 13:

a. Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers. Consider the R-linear map $\varphi : R^3 \longrightarrow R^2 : m \mapsto A \cdot m$ where

$$A = \begin{pmatrix} 1 + x^4 - x^7 + 3x^{100} & \cos(x) & 2 - \exp(x) \\ x^4 - 5x^8 & \sum_{i=0}^{\infty}(5x + x^2)^i & 0 \end{pmatrix} \in \mathrm{Mat}(2 \times 3, R).$$

Is φ an epimorphism?

b. Let $p \in \mathbb{Z}$ be a prime number. Consider the subring $R = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \not| b\right\} \leq \mathbb{Q}$ of the rational numbers, and consider $M = \mathbb{Q}$ as an R-module.

(1) Show that R is local with maximal ideal $\mathfrak{m} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \not| b, p \mid a\right\}$.

(2) $\mathfrak{m} \cdot M = M$, but $M \neq 0$.

(3) Find a discrete set of generators for M.

Exercise 14: Let R be a ring and P an R-module. Show that the following statements are equivalent:

a. If $\varphi \in \mathrm{Hom}_R(M, N)$ is surjective and $\psi \in \mathrm{Hom}_R(P, N)$, then there is a $\alpha \in \mathrm{Hom}_R(P, M)$ such that $\varphi \circ \alpha = \psi$, i.e.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\, \alpha"'] \arrow[d, "\psi"] \\ M \arrow[r, two heads, "\varphi"'] & N \end{tikzcd}$$

b. If $\varphi \in \mathrm{Hom}_R(M, N)$ is surjective, then $\varphi_* : \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_R(P, N) : \alpha \mapsto \varphi \circ \alpha$ is surjective.

c. If $0 \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow 0$ is exact, then it is split exact.

d. There is free module F and a submodule $M \leq F$ such that $P \oplus M \cong F$.

Exercise 15: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Show, if M' and M'' are finitely generated, then so is M.

Hint, you can do the proof using the Snake Lemma and the fact that a free module is projective. Alternatively you can simply write down a set of generators.

Exercise 16: Let R be a ring, M, M' and M'' R-modules, $\varphi \in \mathrm{Hom}_R(M', M)$ and $\psi \in \mathrm{Hom}_R(M, M'')$.
Show that

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

is exact if and only if for all R-modules P the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', P) \xrightarrow{\psi^*} \mathrm{Hom}_R(M, P) \xrightarrow{\varphi^*} \mathrm{Hom}_R(M', P)$$

is exact.

In-Class Exercise 9: Consider $R = \mathbb{R}[x, y, z]$ and $M = \langle xy, xz, yz \rangle$. Find a polynomial $F \in R[t]$ such that $F(\varphi) = 0$ where φ is the restriction to M of the map

$$R \longrightarrow R : f \mapsto f \cdot (x + y + z).$$

In-Class Exercise 10: Let $R = K[x, y]$ and $I = \langle x, y \rangle$. Find R-linear maps such that the following sequence is an exact sequence of R-linear maps:

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$