Fachbereich Mathematik | Winter Semester 2014/15, Set 5
Thomas Markwig | Cornelia Rottner

Commutative Algebra

Due date: Monday, 01/12/2014, 10h00

Exercise 17: Suppose that $(R, \mathfrak{m})$ is local ring and that $M \oplus R^m \cong R^n$ for some $n \geq m$. Show that then $M \cong R^{n-m}$.

Exercise 18: Let R' be an R-algebra and M and N be R-modules. Show that there is an isomorphism of R'-modules

$$\Phi : \big(M \otimes_R N\big) \otimes_R R' \longrightarrow \big(M \otimes_R R'\big) \otimes_{R'} \big(N \otimes_R R'\big) : m \otimes n \otimes r' \mapsto (m \otimes r') \otimes (n \otimes 1).$$

Recall that $M \otimes_R R'$ is an R'-module via $r' \cdot (m \otimes s') := m \otimes (r' \cdot s')$.

Exercise 19: Let $(R, \mathfrak{m})$ be a local ring, and M and N be finitely generated R-modules. Show that $M \otimes N = 0$ if and only if $M = 0$ or $N = 0$.

Hint, use Exercise 18 and Nakayama's Lemma.

Exercise 20: Let R be a ring, M and N be R-modules, and suppose $N = \langle n_\lambda \mid \lambda \in \Lambda \rangle$. Show:

a. $M \otimes_R N = \left\{ \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \;\middle|\; m_\lambda \in M \text{ and only finitely many } m_\lambda \neq 0 \right\}$.

b. Let $x = \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \in M \otimes_R N$ with $m_\lambda \in M$ and only finitely many $m_\lambda \neq 0$.

   Then $x = 0$ if and only if there exist $m'_\theta \in M$ and $a_{\lambda,\theta} \in R$, $\theta \in \Theta$ some index set, such that

$$m_\lambda = \sum_{\theta \in \Theta} a_{\lambda,\theta} \cdot m'_\theta \quad \text{for all} \quad \lambda \in \Lambda$$

   and

$$\sum_{\lambda \in \Lambda} a_{\lambda,\theta} \cdot n_\lambda = 0 \quad \text{for all} \quad \theta \in \Theta.$$

Hint, for part b. consider first the case that N is free in the $(n_\lambda \mid \lambda \in \Lambda)$ and show that in that case actually all m_λ are zero. Then consider a free presentation $\bigoplus_{\theta \in \Theta} R \to \bigoplus_{\lambda \in \Lambda} R \to N \to 0$ of N and tensorize this with M.

In-Class Exercise 11:

a. Consider the $\mathbb{Z}$-modules $M = \mathbb{Z}/2\mathbb{Z}$ and $N = \mathbb{Z}/4\mathbb{Z}$. How many elements does $M \otimes_{\mathbb{Z}} N$ have? Is it isomorphic to a $\mathbb{Z}$-module that you know?

b. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ and the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$. What is its dimension?

In-Class Exercise 12: Let K be a field. Is the K-vector space $K[x] \otimes_K K[y]$ isomorphic to a K-vector space that you know very well? Can you define a multiplication on the tensor product, such that it becomes a K-algebra that you know?