Fachbereich Mathematik | Winter Semester 2014/15, Set 6
Thomas Markwig | Cornelia Rottner

Commutative Algebra

Due date: Monday, 08/12/2014, 10h00

Exercise 21: Let $S \subseteq R$ be a multiplicatively closed subset, and consider the ring extension $i : R \longrightarrow S^{-1}R : r \mapsto \frac{r}{1}$. Show that

$$\{P \in \operatorname{Spec}(R) \mid S \cap P = \emptyset\} \longrightarrow \operatorname{Spec}\left(S^{-1}R\right) : P \mapsto P^e = S^{-1}P$$

is bijective with inverse

$$\operatorname{Spec}\left(S^{-1}R\right) \longrightarrow \{P \in \operatorname{Spec}(R) \mid S \cap P = \emptyset\} : Q \mapsto Q^c = i^{-1}(Q).$$

In particular, for prime ideals $P \in \operatorname{Spec}(R)$ we have $(P^e)^c = P$.

Exercise 22:

a. Let K be a field, $R = K[x,y,z]/\langle xz, yz\rangle$ and $P = \langle x, y, z-1\rangle \trianglelefteq R$. Show $R_P \cong K[z]_{\langle z-1\rangle}$.

b. Let R be a ring, $f \in R$ a non-zero-divisor. Show $R_f \cong R[x]/\langle fx - 1\rangle$.

Exercise 23: Let R be a ring and $\mathcal{N}(R)$ its nilradical. Show:

a. If $S \subseteq R$ multiplicatively closed, then $\mathcal{N}(S^{-1}R) = S^{-1}\mathcal{N}(R)$.

b. A ring is called *reduced* if it has no nilpotent elements except 0. Show that "being reduced" is a local property, i.e. the following are equivalent:

   (1) R is reduced.

   (2) R_P is reduced for each $P \in \operatorname{Spec}(R)$.

   (3) $R_{\mathfrak{m}}$ is reduced for each $\mathfrak{m} \lhd \cdot R$.

c. Show that "being flat" is a local property, i.e. if M is an R-module, then the following are equivalent:

   (1) M is a flat R-module.

   (2) M_P is a flat R_P module for each $P \in \operatorname{Spec}(R)$.

   (3) $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$ module for each $\mathfrak{m} \lhd \cdot R$.

Hint for part c., use Exercise 21 and note that any R_P-module N is also an R-module and that $N_P = N$.

Exercise 24: Let $I := \langle 2, 1+\sqrt{-5}\rangle \lhd \mathbb{Z}\left[\sqrt{-5}\right]$. Show that I as an R-module is projective, but not free.

Hint, note that $2 \in I \cdot I$. Use this to show that $I \neq \langle x\rangle$ for any x, while for any prime P containing I we have I_P is generated by $1+\sqrt{-5}$. For the last statement use Nakayama's Lemma in a sensible way!

In-Class Exercise 13: Let $R = K[x,y]$ and $P = K[x,y,z]/\langle xz - x, yz - y - z + 1\rangle$. Is P a flat R-module?

In-Class Exercise 14: Let $\mathfrak{m} = \langle x, y\rangle \lhd K[x,y]$. Localize the ring R and the module P in In-Class Exercise 13 at $\mathfrak{m}$. Is the resulting module $P_{\mathfrak{m}}$ a flat $R_{\mathfrak{m}}$-module?