Fachbereich Mathematik — Winter Semester 2014/15, Set 7
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Commutative Algebra

Due date: Monday, 15/12/2014, 10h00

Exercise 25: Let M be an R-module and $\varphi : M \to M$ an R-linear map. Show:

a. If M is noetherian and φ is surjective, then φ is an isomorphism.

b. If M is artinian and φ is injective, then φ is an isomorphism.

Hint, consider the kernel respectively cokernel of φ^n for $n \in \mathbb{N}$.

Exercise 26: Which of the following rings R_i is noetherian?

a. $R_1 = \left\{ \frac{g}{h} \in \operatorname{Quot}(\mathbb{C}[x]) \;\middle|\; h(z) \neq 0 \text{ for } |z| = 1 \right\}$.

b. $R_2 = \left\{ f \in \mathbb{C}\{x\} \;\middle|\; f \text{ has infinite radius of convergence} \right\}$.

c. $R_3 = \left\{ f \in \mathbb{C}[x] \;\middle|\; \frac{\partial^i f}{\partial x^i}(0) = 0 \text{ for } i = 1, \ldots, k \right\}$, k fixed.

Exercise 27: Let $\mathbb{Q} \subseteq K$ be a field extension such that $\dim_{\mathbb{Q}}(K) < \infty$, and suppose R is a subring of K containing $\mathbb{Z}$ such that $I \cap \mathbb{Z} \neq \{0\}$ for each ideal $0 \neq I \trianglelefteq R$. Show that R is noetherian.

Hint, show first that $\dim_{\mathbb{Z}/p\mathbb{Z}}(R/pR) \leq \dim_{\mathbb{Q}}(K)$ for any prime number p. Then conclude that for $0 \neq m \in I \cap \mathbb{Z}$ the set R/mR (and hence I/mR) is finite by induction on the number of prime factors of $m = p_1 \cdots p_k$, p_i prime number. – Remark: using a bit field theory one can show that the assumption $I \cap \mathbb{Z} \neq \{0\}$ is always fulfilled.

Exercise 28: Let $R \subseteq R' \subseteq R''$ be rings, $R'' = R[a_1, \ldots, a_n]$ a finitely generated R-algebra and R'' finitely generated as an R'-module. Show, if R noetherian, then R' is finitely generated as an R-algebra and noetherian.

Recall, $R[a_1, \ldots, a_n] = \{f(a_1, \ldots, a_n) \mid f \in R[x_1, \ldots, x_n]\}$ is the set of all polynomial expressions in $a_1, \ldots, a_n$ with coefficients in R. Hint, if $R'' = \langle b_1, \ldots, b_m \rangle_{R'}$, then write a_i and $b_i \cdot b_j$ as linear combinations of the b_ν and consider the R-algebra generated by the coefficients.

In-Class Exercise 15: Show that $\mathbb{C}[x, y]/I$ with $I = \langle x^3 - x^2, x^2y + 2x^2, xy - y, y^2 + 2y \rangle$ is an artinian ring and decompose it as a direct sum of two local artinian rings.

In-Class Exercise 16: Is $K(x) = \operatorname{Quot}(K[x])$ a noetherian $K[x]_{\langle x \rangle}$-module?