Fachbereich Mathematik
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Winter Semester 2014/15, Set 8

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Commutative Algebra

Due date: Monday, 05/01/2015, 10h00

Exercise 29:

a. Let $\varphi : R \to R'$ be a ring homomorphism and $Q \lhd R'$ a P-primary ideal. Show that $Q^c = \varphi^{-1}(Q)$ is $P^c = \varphi^{-1}(P)$-primary.

b. Let R be a ring, $P \in \text{Spec}(R)$, and $n \geq 1$. Show that the *symbolic power* $P^{(n)} := \{a \in R \mid \exists\, s \in R \setminus P \,:\, s \cdot a \in P^n\}$ is a P-primary ideal.

Note, if $\iota : R \to R_P : a \mapsto \frac{a}{1}$, then $P^{(n)} = \big((P^n)^e\big)^c = \iota^{-1}\big(\langle P^n \rangle_{R_P}\big)$.

Exercise 30: Let R be a integral domain of dimension $\dim(R) = 1$, and let $0 \neq I \unlhd R$.

a. Show if $I = Q_1 \cap \ldots \cap Q_n$ is a minimal primary decomposition, then $I = Q_1 \cdots Q_n$.

b. If R is noetherian, then every non-zero ideal I is a finite product of primary ideals Q_i with $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$, and the factors are unique up to ordering.

Hint, Chinese Remainder Theorem.

Exercise 31: Find a minimal primary decomposition of $I = \langle 6 \rangle \lhd \mathbb{Z}\big[\sqrt{-5}\big]$.

Hint, consider the ideals $P = \langle 2, 1 + \sqrt{-5} \rangle$, $Q = \langle 3, 1 + \sqrt{-5} \rangle$, and $Q' = \langle 3, 1 - \sqrt{-5} \rangle$.

Exercise 32: Let $R = K[x, y, z]$ for some field K.

a. Let $P = \langle x, y \rangle$ and $Q = \langle y, z \rangle$. Calculate a minimal primary decomposition of $I = P \cdot Q$. Which of the components are isolated, which are embedded?

b. Calculate a primary decomposition of $J = \langle xz - y^2, y - x^2 \rangle$.

Hint, in part b. consider $\varphi : R \to K[x]$ with $x \mapsto x, y \mapsto x^2, z \mapsto x^3$, $P = \ker(\varphi)$, and $Q = \langle x, y \rangle$. Show that $\ker(\varphi) = \langle y - x^2, z - x^3 \rangle$ using division with remainder.

In-Class Exercise 17: Find the primary decomposition of $\langle x^3y^2 - xy^4 \rangle$ in $\mathbb{K}[x, y]$.

In-Class Exercise 18: Find the primary decomposition of $\langle x^2 - x, xy - x \rangle$ in $K[x, y]$.

In-Class Exercise 19: Find the primary decomposition of the ideal $\langle x^3 - x^2 - x + 1, x^2y - x^2 - 2xy + 2x + y - 1, xy + y, y^2 - y \rangle$ in $K[x, y]$ and in $K[x, y]_{x+1}$.