

## Commutative Algebra

Due date: Monday, 19/01/2015, 10h00

**Exercise 39 a., b. and c. need NOT be handed in for marking.**

**Exercise 37:** Let  $R \subset R'$  be an integral ring extension and let  $\mathfrak{m}' \triangleleft R'$  be a maximal ideal such that  $\mathfrak{m} = \mathfrak{m}' \cap R \triangleleft R$  is maximal as well. Is then  $R'_{\mathfrak{m}'}$  integral over  $R_{\mathfrak{m}}$ ?

Hint, consider  $R = K[x^2 - 1]$ ,  $R' = K[x]$ ,  $\mathfrak{m}' = \langle x - 1 \rangle$ , and  $f = \frac{1}{1+x} \in R'_{\mathfrak{m}'}$ .

**Exercise 38:** Let  $R \subset R'$  be integral domains and  $f, g \in R'[x]$  be monic polynomials. Show that if  $f \cdot g \in \text{Int}_{R'}(R)[x]$ , then  $f, g \in \text{Int}_{R'}(R)[x]$ .

Note, if we apply this to  $R = \mathbb{Z}$  and  $R' = \mathbb{Q}$ , then we get for monic  $f, g \in \mathbb{Q}[x]$  that  $f \cdot g \in \mathbb{Z}[x]$  implies  $f, g \in \mathbb{Z}[x]$ .

**Exercise 39: [Rings of Integers of Quadratic Number Fields]**

Let  $d \in \mathbb{Z} \setminus \{0, 1\}$  be a squarefree number (i.e. no square  $a^2$  divides  $d$ ), then  $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$  is a field extension of  $\mathbb{Q}$  with  $\dim_{\mathbb{Q}} \mathbb{Q}[\sqrt{d}] = 2$ . Consider the conjugation

$$C : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q}[\sqrt{d}] : a + b\sqrt{d} \mapsto a - b\sqrt{d},$$

the norm

$$N : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) \cdot C(a + b\sqrt{d}) = a^2 - b^2d,$$

and the trace

$$T : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q} : a + b\sqrt{d} \mapsto (a + b\sqrt{d}) + C(a + b\sqrt{d}) = 2a.$$

Show:

- $C(x \cdot y) = C(x) \cdot C(y)$  and  $N(x \cdot y) = N(x) \cdot N(y)$  for  $x, y \in \mathbb{Q}[\sqrt{d}]$ .
- $C$  and  $T$  are  $\mathbb{Q}$ -linear.
- If  $x \in \mathbb{Q}[\sqrt{d}] \setminus \mathbb{Q}$ , then  $\mu_x = (t - x) \cdot (t - C(x)) = t^2 - T(x) \cdot t + N(x) \in \mathbb{Q}[t]$  is the minimal polynomial of  $x$  over  $\mathbb{Q}$ .
- $x \in \mathbb{Q}[\sqrt{d}]$  is integral over  $\mathbb{Z}$  if and only if  $T(x)$  and  $N(x)$  are integers.
- $\text{Int}_{\mathbb{Q}[\sqrt{d}]}(\mathbb{Z}) = \mathbb{Z}[\omega_d]$ , where  $\omega_d = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod{4} \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$

**In-Class Exercise 21:** Which of the following ring extensions is integral?

- $K[x] \hookrightarrow K[x, y, z] / \langle z^2 - xy \rangle : x \mapsto \bar{x}$ .
- $K[x] \hookrightarrow K[x, y, z] / \langle z^2 - xy, y^3 - x^2 \rangle : x \mapsto \bar{x}$ .
- $K[x] \hookrightarrow K[x, y, z] / \langle z^2 - xy, x^3 - yz \rangle : x \mapsto \bar{x}$ .