Fachbereich Mathematik — Winter Semester 2014/15, Set 12
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# Commutative Algebra

Due date: Monday, 02/02/2015, 10h00

**Exercise 44:** Prove the Algebraic HNS (Theorem 7.1) using Noether-Normalisation.

**Exercise 45:** Let $R$ be a ring. Show that $\dim(R[x]) \geq \dim(R) + 1$.

Hint, consider ideals of the form $I[x] = \left\{ \sum_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in I \right\}$ for some ideal $I \trianglelefteq R$. – Note, if $R$ is noetherian one can actually show equality, but that is much harder.

**Exercise 46:** Let $R$ be an integral domain. Show:

a. $R$ is a valuation ring if and only if for two ideals $I, J \trianglelefteq R$ we have $I \subseteq J$ or $J \subseteq I$.

b. If $R$ is a valuation ring and $P \in \mathrm{Spec}(R)$, then $R_P$ and $R/P$ are valuation rings.

**Exercise 47: [A valuation on the field $K\{\{t\}\}$]**

Let $K\{\{t\}\}$ be the field from Exercise 3.

a. Show that $\mathrm{ord} : (K\{\{t\}\}^*, *) \longrightarrow (\mathbb{R}, +) : f \mapsto \min\{\alpha \in \mathbb{R} \mid f(\alpha) \neq 0\}$ is a valuation.

b. $R_{\mathrm{ord}}$ is not noetherian, hence ord is not discrete, but $\dim(R_{\mathrm{ord}}) = 1$.

c. If $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ are algebraically independent over $\mathbb{Q}$, then $(t^{\alpha_1}, \ldots, t^{\alpha_n})$ are algebraically independent over $K$. In particular, $\mathrm{trdeg}_K \left(K\{\{t\}\}\right) = \infty$.

Hint for part b., note that $\mathfrak{m}_{R_{\mathrm{ord}}} = \langle t^\alpha \mid \alpha > 0 \rangle$, where $t^\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $t^\alpha(\alpha) = 1$ and $t^\alpha(\beta) = 0$ for $\beta \neq \alpha$.

**In-Class Exercise 24:** Compute the dimension of $K[x, y, z]_{\langle x^2 - yz \rangle}$.

**In-Class Exercise 25:** Is the ring $\mathbb{C}[x, y]/\langle x^2 - y^2 - y^3 \rangle$ a Dedekind domain?

**In-Class Exercise 26:** Is the ring $\mathbb{C}[x, y]/\langle x - y^2 - y^3 \rangle$ a Dedekind domain?