Commutative Algebra

In class exercise 1: We call an ideal I in the polynomial ring $K[\underline{x}] = K[x_1, \ldots, x_n]$ a *monomial ideal* if I is generated by (possibly infinitely many) monomials.
Given two monomials $\underline{x}^\alpha$ and $\underline{x}^\beta$ we say that $\underline{x}^\alpha$ *divides* $\underline{x}^\beta$ if there is a monomial $\underline{x}^\gamma$ such that $\underline{x}^\alpha \cdot \underline{x}^\gamma = \underline{x}^\beta$, i.e. $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, n$.
And we define the *least common multiple* of $\underline{x}^\alpha$ and $\underline{x}^\beta$ in the obvious way as

$$\operatorname{lcm}\left(\underline{x}^\alpha, \underline{x}^\beta\right) = x_1^{\max\{\alpha_1,\beta_1\}} \cdots x_n^{\max\{\alpha_n,\beta_n\}},$$

i.e. it is the monomial of lowest degree which is divisible by both monomials.

a. Show that for an ideal I the following are equivalent:
   (1) I is a monomial ideal.
   (2) For any $f \in I$ also all monomials occuring in f belong to I.
   (3) There is a generating set B of I such that for any $f \in B$ all monomials of f belong to I.

b. If $I = \langle \underline{x}^\alpha \mid \alpha \in \Lambda \rangle$ and $\underline{x}^\beta \in I$ then there is an $\alpha \in \Lambda$ such that $\underline{x}^\alpha$ divides $\underline{x}^\beta$.

c. Let $I = \langle \underline{x}^\alpha \mid \alpha \in \Lambda \rangle$ and $J = \langle \underline{x}^\beta \mid \beta \in \Lambda' \rangle$ be two monomial ideals in $K[\underline{x}]$. Show that

$$I \cap J = \left\langle \operatorname{lcm}\left(\underline{x}^\alpha, \underline{x}^\beta\right) \;\middle|\; \alpha \in \Lambda, \beta \in \Lambda' \right\rangle$$

and

$$I : \langle \underline{x}^\gamma \rangle = \left\langle \frac{\operatorname{lcm}(\underline{x}^\alpha, \underline{x}^\gamma)}{\underline{x}^\gamma} \;\middle|\; \alpha \in \Lambda \right\rangle.$$

Hint for part c., show first that the two ideals are monomial ideals.

In class exercise 2: We will now introduce some basic commands for SINGULAR.
In SINGULAR we can work with two types of rings that we have introduced so far in the lecture, polynomial rings $K[x_1, \ldots, x_n]$ and power series rings $K[[x_1, \ldots, x_n]]$.
The polynomial ring $\mathbb{Q}[x, y, z]$ is defined in SINGULAR as:

```
ring r=0,(x,y,z),dp;
```

Here, `0` stands for the characteristic of $\mathbb{Q}$ and `dp` says that we are working with a **p**olynomial ring.
The power series ring $\mathbb{Z}/5\mathbb{Z}[[x_1, \ldots, x_4]]$ is defined in SINGULAR as:

```
ring r=5,(x(1..4)),ds;
```

Here, 5 stands for the characteristic of $\mathbb{Z}/5\mathbb{Z}$ and `dp` says that we are working with a power **s**eries ring — actually this is not quite true, but morally it is, and we need the notion of *localisation* to be more precise.

Once we have fixed a ring we can define polynomials and ideals and perform operations with them:

```
LIB "all.lib";        // load libraries needed e.g. for the radical
ring r=0,(x,y,z),dp;
poly f=x^3*y+5*z^2;
poly g=3x2y-xz2;     // this is short hand for 3*x^2*y-x*z^2
ideal I=f,g,x2y;
ideal J=x+y;
I*J;                 // the product of I and J
intersect(I,J);      // intersect the two ideals
quotient(I,J);       // compute the ideal quotient
radical(I);          // compute the radical of I
I=std(I);            // replace the generators of I by better ones
reduce(f,I);         // test if f belongs to I
reduce(J,I);         // test if J is contained in I
```

Consider the ideal $I = \langle x^2y^5, x^6, y^2\rangle$ and $J = \langle x^2y, xy^4\rangle$. Compute the following ideals with SINGULAR:

a. $I \cap J$.

b. $I \cdot J$.

c. $I : \langle x^3y^6\rangle$.

d. $\sqrt{I}$.

e. Test if the polynomial $x^7 + xy^8$ is in I.

Verify the results without SINGULAR.

In class exercise 3: Which of the following ideals are monomial ideals?

a. $I = \langle x^2y - y^3, x^3\rangle \lhd \mathbb{Q}[x, y, z]$.

b. $I = \langle x^4 - x^2y^2 + y^4, 2x^3 - xy^2, 2y^3 - x^2y\rangle \lhd \mathbb{Q}[x, y, z]$

c. $I = \langle x^{12}y^7 + x^9y + xyz^3 + yz^3, x^8 - xyz, yz^3, x^8 - yz^3, x^{12}y^7\rangle \lhd \mathbb{Q}[x, y, z]$.