Commutative Algebra

Submit by: Monday, 25/10/21, 10 am

Exercise 1: Let $0 \neq f = \sum_{|\alpha|=0}^{m} a_\alpha \underline{x}^\alpha \in R[x_1, \ldots, x_n]$ be a polynomial over the ring R. Recall:

$$\deg(f) := \max\{|\alpha| \mid a_\alpha \neq 0\}$$

is the *degree* of f, and we set $\deg(0) = -\infty$. Show for $f, g \in R[x_1, \ldots, x_n]$:

a. $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$,

b. $\deg(f \cdot g) \leq \deg(f) + \deg(g)$,

c. $\deg(f \cdot g) = \deg(f) + \deg(g)$, if R is an integral domain.

Note, R is an integral domain if $r \cdot r' = 0$ for $r, r' \in R$ implies that $r = 0$ or $r' = 0$.

Exercise 2: Let K be a ring, $d \in \mathbb{N}$, and

$$K[x_1, \ldots, x_n]_d = \left\{ \sum_{|\alpha|=\alpha_1+\ldots+\alpha_n=d} a_\alpha \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \;\middle|\; a_\alpha \in K \right\}.$$

We call the elements of $K[x_1, \ldots, x_n]_d$ homogeneous of degree d.

a. Show that every polynomial $0 \neq f \in K[x_1, \ldots, x_n]$ of degree d admits a unique decomposition $f = f_0 + \ldots + f_d$ with $f_i \in K[x_1, \ldots, x_n]_i$. We call the f_i the *homogeneous summands* of f.

b. An ideal $I \trianglelefteq K[x_1, \ldots, x_n]$ is called *homogeneous*, if $f \in I$ implies that the homogeneous summands of f belong to I.

   Show that I is homogeneous if and only if I is generated by homogeneous elements.

In class exercise 1: [The field $K\{\{t\}\}$]

a. We call $A \subset \mathbb{R}$ *suitable* if A is infinite countable, bounded from below, and has no limit point, and we then set $\mathcal{A} := \{A \subset \mathbb{R} \mid A \text{ is suitable}\}$. Show that for $A, B \in \mathcal{A}$

$$A + B := A \cup B \in \mathcal{A} \quad \text{and} \quad A * B := \{a + b \mid a \in A, b \in B\} \in \mathcal{A}.$$

b. Let K be any field and consider the set

$$K\{\{t\}\} := \{f : \mathbb{R} \to K \mid \exists\, A \in \mathcal{A} \;:\; f(\alpha) = 0 \;\forall\, \alpha \notin A\}.$$

We define two binary operations on $K\{\{t\}\}$:

$$f + g : \mathbb{R} \to K : \alpha \mapsto f(\alpha) + g(\alpha)$$

and

$$f * g : \mathbb{R} \to K : \alpha \mapsto \sum_{\gamma \in \mathbb{R}} f(\alpha - \gamma) \cdot g(\gamma),$$

note for the latter that for a fixed α only finitely many summands are non-zero!

Show that $(K\{\{t\}\}, +, *)$ is a field.

Hint for part b., show first that $(K\{\{t\}\}, +)$ is a subgroup of $(K^{\mathbb{R}}, +)$. The hard part is to show that every non-zero element of $K\{\{t\}\}$ has an inverse. For this consider first the case that $f(\alpha) = 0$ for $\alpha < 0$ and $f(0) = 1$, and use the geometric series.

Remark 1

Let $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a sequence of real numbers. We define

$$\alpha_n \nearrow \infty \quad :\Longleftrightarrow \quad (\alpha_n)_{n\in\mathbb{N}} \text{ is strictly monotonously increasing and unbounded,}$$

and we set $\mathbb{A} := \{(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \alpha_n \nearrow \infty\}$. Obviously,

$$\Phi : \mathbb{A} \longrightarrow \mathcal{A} : (\alpha_n)_{n\in\mathbb{N}} \mapsto \{\alpha_n \mid n \in \mathbb{N}\}$$

is bijective.

For $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{A}$ and $(a_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ we define

$$\sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n} : \mathbb{R} \longrightarrow K : \alpha \mapsto \begin{cases} a_n, & \text{if } \alpha = \alpha_n, \\ 0, & \text{else.} \end{cases}$$

That is, we use the “series” in order to store the values of a funciton in such a way, that the value at α_n is just the coefficient at t^{α_n}. Thus

$$\begin{aligned} K\{\{t\}\} &= \Big\{f : \mathbb{R} \to K \;\Big|\; \exists\, \alpha_n \nearrow \infty \;:\; f(\alpha) = 0 \;\forall\, \alpha \notin \{\alpha_n \mid n \in \mathbb{N}\}\Big\} \\ &= \left\{ \sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n} \;\middle|\; \alpha_n \nearrow \infty, a_n \in K \right\}. \end{aligned}$$

Given $f = \sum_{n=0}^{\infty} a_n \cdot t^{\alpha_n}, g = \sum_{n=0}^{\infty} b_n \cdot t^{\beta_n} \in K\{\{t\}\}$.

a. $f = g$ if and only if $a_n = b_m$ whenever $\alpha_n = \beta_m$ and if $a_i = b_j = 0$ if there is no matching.

b. $f * g = \sum_{n=0}^{\infty} \big(\sum_{\alpha_i + \beta_j = \gamma_n} a_i \cdot b_j \big) \cdot t^{\gamma_n}$, where $(\gamma_n)_{n\in\mathbb{N}} = \Phi^{-1}\big(\Phi((\alpha_n)_{n\in\mathbb{N}}) * \Phi((\beta_n)_{n\in\mathbb{N}})\big)$.

c. $f + g = \sum_{n=0}^{\infty} \big(f(\gamma_n) + g(\gamma_n)\big) \cdot t^{\gamma_n}$, where $(\gamma_n)_{n\in\mathbb{N}} = \Phi^{-1}\big(\Phi((\alpha_n)_{n\in\mathbb{N}}) + \Phi((\beta_n)_{n\in\mathbb{N}})\big)$.

d. If $\alpha_0 = 0$ and $a_0 = 1$, then $f^{-1} = \sum_{n=0}^{\infty} \big(- \sum_{k=1}^{\infty} a_k \cdot t^{\alpha_k} \big)^n$.