Fachbereich Mathematik — Winter Semester 2021/22, Set 2
Thomas Markwig — Alheydis Geiger

Commutative Algebra

Submit by: Monday, 01/11/2021, 10 am

The in-class exercises need not be handed in for marking. They should be discussed in class. No rigorous proofs are expected for these.

Exercise 4: Let R be a ring. Obviously $R \hookrightarrow R[x_1, \ldots, x_n] : a \mapsto a$ is a ring homomorphism and thus makes $R[x_1, \ldots, x_n]$ an R-algebra.

a. Show that $R[x_1, \ldots, x_n]$ satisfies the following universal property: if (R', φ) is any R-algebra and $a_1, \ldots, a_n \in R'$ are given, then there is a unique R-algebra homomorphism $\alpha : R[x_1, \ldots, x_n] \to R'$ such that $\alpha(x_i) = a_i$ for all $i = 1, \ldots, n$.

b. Let $I \trianglelefteq R[x_1, \ldots, x_n]$ and $J \trianglelefteq R[y_1, \ldots, y_m]$ and let $\varphi : R[x_1, \ldots, x_n]/I \to R[y_1, \ldots, y_m]/J$. Show that the following are equivalent:

   (a) φ is an R-algebra homomorphism

   (b) There are $f_1, \ldots, f_n \in R[y_1, \ldots, y_m]$ such that $g(f_1, \ldots, f_n) \in J$ for all $g \in I$ and $\varphi(\overline{g}) = \overline{g(f_1, \ldots, f_n)}$ for all $\overline{g} \in R[x_1, \ldots, x_n]/I$.

   (c) There is an R-algebra homomorphism $\psi : R[x_1, \ldots, x_n] \to R[y_1, \ldots, y_m]$ such that $\psi(I) \subseteq J$ and $\varphi(\overline{g}) = \overline{\psi(g)}$.

Note, a. means: we may uniquely define an R-algebra homomorphism on $R[x_1, \ldots, x_n]$ by just specifying the images of the x_i!

Exercise 5: Let R be a ring and $I, J_1, \ldots, J_n \trianglelefteq R$. Show that:

a. $I : \left(\sum_{i=1}^n J_i\right) = \bigcap_{i=1}^n (I : J_i)$.

b. $\left(\bigcap_{i=1}^n J_i\right) : I = \bigcap_{i=1}^n (J_i : I)$.

c. $\sqrt{J_1 \cap \ldots \cap J_n} = \sqrt{J_1} \cap \ldots \cap \sqrt{J_n}$.

d. $\sqrt{J_1 + \ldots + J_n} \supseteq \sqrt{J_1} + \ldots + \sqrt{J_n}$.

Exercise 6: Let R be a ring and $f = \sum_{n=0}^\infty a_n x^n \in R[[x]]$ a formal power series over R. Show:

a. f is a *unit* if and only if a_0 is a unit in R.

b. What are the units in $K[[x]]$ if K is a field?

c. x is not a zero-divisor in $R[[x]]$.

d. If f is nilpotent, then a_n is nilpotent for all n. Is the converse true?

Hint for a., consider first the case $a_0 = 1$ and recall that $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$.

In class exercise 4: Consider the ring extension

$$\iota : \mathbb{Z} \longrightarrow \mathbb{Z}_7 = \left\{ \frac{z}{7^n} \;\middle|\; n \geq 0, z \in \mathbb{Z} \right\} : z \mapsto z$$

and the ideals $I = \langle 84 \rangle \lhd \mathbb{Z}$ and $J = \langle 15 \rangle \lhd \mathbb{Z}_7$. Give generators I^e, I^{ec}, J^c, and J^{ce}.

In class exercise 5: Does the following equality of ideals hold in the polynomial ring $\mathbb{C}[x, y]$:

$$\langle x^3 - x^2, x^2y - x^2, xy - y, y^2 - y \rangle = \langle x^2, y \rangle \cap \langle x - 1, y - 1 \rangle.$$

In class exercise 6: What are the prime ideals in $\mathbb{C}[x, y]$ containing the ideal $I = \langle x^2y - x^2 \rangle$.