Fachbereich Mathematik Winter Semester 2021/22, Set 3
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Commutative Algebra

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Exercise 8: Let R be a ring such that for every $r \in R$ there is an $n = n(r) > 1$ such that $r^n = r$.

a. Show that $\mathrm{Spec}(R) = \mathfrak{m} - \mathrm{Spec}(R)$.

b. Give an example of such a ring R which is not a field.

Exercise 9: Let R be a ring and $N(R)$ its nil-radical. Show the following are equivalent:

a. $R/N(R)$ is a field.

b. $|\mathrm{Spec}(R)| = 1$.

c. Every element of R is either a unit or nilpotent.

Give an example for such a ring which is not a field.

Exercise 10: Let M be an R-module.

a. Prove that $\mu : M \to \mathrm{Hom}_R(R, M)$ with $\mu(m) : R \to M : r \mapsto r \cdot m$ is an isomorphism.

b. Give an example where $M \not\cong \mathrm{Hom}_R(M, R)$.

Exercise 11: Let R be an integral domain and $0 \neq I \trianglelefteq R$.
Show that I as R-module is free if and only if I is principal.

In class exercise 7: Which of the following ideals I in $\mathbb{Z}[x]$ is a maximal ideal?

a. $I = \langle 5, 11x^3 + x - 1 \rangle$.

b. $I = \langle 4, x^2 + x + 1, x^2 + x - 1 \rangle$.

How many elements does the corresponding field $\mathbb{Z}[x]/I$ have?

In class exercise 8: Let K be any field. Show that $x^2 - y^3 \in K[x, y]$ is irreducible.