

Commutative Algebra

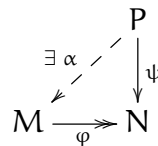
Submit by: Monday, 15/11/2021, 10 am

Exercise 12: Let $p \in \mathbb{Z}$ be a prime number. Consider the subring $R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b \right\} \subseteq \mathbb{Q}$ of the rational numbers, and consider $M = \mathbb{Q}$ as an R -module.

- a. Show that R is local with maximal ideal $\mathfrak{m} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b, p \mid a \right\}$.
- b. $\mathfrak{m} \cdot M = M$, but $M \neq 0$.
- c. Find a *small* set of generators for M .

Exercise 13: Let R be a ring and P an R -module. Show that the following statements are equivalent:

- a. If $\varphi \in \text{Hom}_R(M, N)$ is surjective and $\psi \in \text{Hom}_R(P, N)$, then there is an $\alpha \in \text{Hom}_R(P, M)$ such that $\varphi \circ \alpha = \psi$, i.e.



- b. If $\varphi \in \text{Hom}_R(M, N)$ is surjective, then $\varphi_* : \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N) : \alpha \mapsto \varphi \circ \alpha$ is surjective.
- c. If $0 \rightarrow L \rightarrow M \rightarrow P \rightarrow 0$ is exact, then it is split exact.
- d. There is free module F and a submodule $M \leq F$ such that $P \oplus M \cong F$.

We call such a module *projective*.

Exercise 14: Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules. Show, if M' and M'' are finitely generated, then so is M .

Hint, you can do the proof using the Snake Lemma and the fact that a free module is projective. Alternatively you can simply write down a set of generators.

In class exercise 9: Consider $R = \mathbb{R}[x, y, z]$ and $M = \langle xy, xz, yz \rangle$. Find a polynomial $F \in \mathbb{R}[t]$ such that $F(\varphi) = 0$ where φ is the restriction to M of the map

$$R \longrightarrow R : f \mapsto f \cdot (x + y + z).$$

In class exercise 10: Let $R = \mathbb{R}[[x]]$ be the ring of formal power series over the real numbers. Consider the R -linear map $\varphi : R^3 \rightarrow R^2 : m \mapsto A \cdot m$ where

$$A = \begin{pmatrix} 1 + x^4 - x^7 + 3x^{100} & \cos(x) & 2 - \exp(x) \\ x^4 - 5x^8 & \sum_{i=0}^{\infty} (5x + x^2)^i & 0 \end{pmatrix} \in \text{Mat}(2 \times 3, R).$$

Is φ an epimorphism?

In class exercise 11: Let $R = K[x, y]$ and $I = \langle x, y \rangle$. Find R -linear maps such that the following sequence is an exact sequence of R -linear maps:

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$